

## FOREWORD TO THE ELEMENTS OF EUCLID

Legend has it that Plato put up over the entrance to the Academy the words: Let no one ignorant of geometry come in here. The Philosopher also in his Proemium to Wisdom, in giving the order of wonder, also includes the wonder about the incommensurability of the diagonal and side of a square. And mathematics, as found in Euclid's *Elements*, is one of the three parts of looking philosophy which especially begins in wonder. And among the parts of looking philosophy, the mathematical sciences have an excellence of certitude and clarity. Moreover, Aristotle also states in the thirteenth book of Wisdom or First Philosophy, the thirteenth book *meta* (after) *ta* (the) *phusika* (the books in natural philosophy) that the mathematical sciences especially show the main forms of the beautiful.

A complete foreword to a book should unfold the matter, end, form and author of that book. The matter of a book is what it is about. The end of the book is the reason(s) why knowledge of that matter is sought. The form of the book is the order in which the matter is considered and how it is considered. The author is, of course, who wrote it. The title of the book is explained under one or more of these.

### THE MATTER OF THE ELEMENTS

The matter of the *Elements* is both that of geometry, which is a reasoned out knowledge of lines, angles, and especially plane and solid figures (in general, continuous quantity) and also that of arithmetic, which is a reasoned out knowledge of numbers.

### THE END OF THE ELEMENTS

The word *mathematician* comes from the Greek word meaning one fond of learning. And this is a sign that knowledge of the above matter was sought as itself the end. Hence, mathematics is one of the three theoretical or looking philosophies distinguished by Aristotle in the sixth book of the *Metaphysics*. The word

*theoretical* comes from the Greek word for *looking* and looking is for the sake of seeing. Hence, we sometimes speak of *theorems* in geometry for these are things to look at and reason delights in looking at them. And the clarity and certitude with which such things can be known makes the knowledge of them in some way better than every other reasoned out knowledge. (We say *in some way* because, simply or without qualification, knowledge of better things is better knowledge, as Aristotle teaches us in the *Parts of Animals*.)

The above end or purpose of the *Elements* may surprise the slaves of modern customs who think of mathematics as only a practical tool for calculating.

## THE FORM OF THE ELEMENTS

The *form* of the *Elements* is both the *order* in which its matter is considered and *how* that matter is considered.

Under the order of the *Elements*, we should consider the division and order of its books and the order in which things are considered even in the same book (as well as in the order of the books).

The *Elements* as a whole seem to be divided into four parts. The first part, Books 1-6, is plane geometry, the reasoned out knowledge of lines, plane angles and especially plane figures (such as triangle, square and circle). The second part, Books 7-9, is a reasoned out knowledge of numbers. The third part, Book 10, is about the irrational or continuous quantities that do not have to each other the ratio of a number to a number. And the fourth part, Books 11-13, is solid geometry, the reasoned out knowledge of solid angles and especially solid figures (such as sphere and cube).

The order in *Euclid's Elements* would seem to be usually the simple before the composed and the equal before the unequal.

The simple before the composed is seen in the whole of geometry in which plane geometry is placed before solid geometry. And within plane geometry, this is also observed. Thus most of the theorems about triangles are found in the first 32 theorems of Book One followed by the theorems about parallelograms. And the fundamental theorems about the parallelogram (Book One, Propositions 33 & 34) are shown through theorems about the triangle. And this order is followed even

within the definitions beginning geometry. Hence, the point is defined first and then line and then surface. And angle is defined before figures such as triangle. And triangle is defined and divided into kinds before quadrilateral is defined and divided into kinds.

Almost the same as this is the equal before the unequal for the most part. What is more equal is defined before what is less equal or unequal. And sometimes the unequal is defined by the equal or by something in its definition. Euclid defines right angles before acute and obtuse angles. And he does not define acute and obtuse like he defines right angles. He does not say that when a straight line meeting a straight line makes unequal angles, the greater is called obtuse and the lesser acute. Rather he defines the obtuse as greater than a right angle and the acute as less than a right angle. And he defines even number before odd number and one definition of odd number includes even number and the other a negation of part of even number's definition.

The equilateral triangle is defined before the isosceles and the latter before the scalene. And among quadrilaterals, the parallelograms (whose opposite sides and angles are equal) are defined before the trapezium whose opposite sides and angles are not. The square is defined first and the rhomboid is defined after the oblong and the rhombus. And the trapezium after the aforesaid parallelograms (whose opposite sides and angles, at least, are equal). And among the first twenty-five theorems or propositions, the equality theorems (1-15) come before the inequality theorems (16-25).

And among the axioms, the equality axioms are given before the inequality axiom (the whole is more than one of its parts).

And in the theorems, the equality theorems come usually before the inequality theorems. And often, in addition, the inequality theorems are proven by the equality theorems. In the rectilinear books, the first one ends with the Pythagorean theorem and its converse. The corresponding inequality theorems for the obtuse triangle and the acute triangle are not given till the end of Book two and both are proven by means of the Pythagorean theorem. And in book One, we prove that angles opposite equal sides are equal (and the converse) before we prove that the angle opposite the greater side is greater than that opposite the lesser side (and its converse). And we prove the latter inequality theorem by the former equality theorem.

In general, both considering the simple before the composed and the equal before the unequal is considering the one before the many. For the simple is more one than the composed. And equality is caused by unity and inequality by multiplicity, as Thomas Aquinas explains (*De Potentia*, Q. 3, Art. 15, corpus):

aequalitas ab unitate causetur, inaequalitas vero ex multitudine quae vario modo se habet (ratione cuius non est aliquid alteri aequale nisi uno modo, inaequale vero secundum multos gradus.)

The question could be raised why the arithmetical books (and also the "irrational" book) are placed between the books of plane geometry and the books of solid geometry. At first sight, it would seem that the books of solid geometry should immediately follow those of plane geometry. It would seem out of order to place books in arithmetic in the middle of geometrical books. Why this is so can be seen fully only by those who have worked their way through these books. But briefly here it can be said that Euclid follows the principle used also by Thomas in ordering the intermediary books on life of Aristotle. Although natural philosophy as a whole proceeds from the general to the particular, Thomas orders the books on sense and the sensible and on memory and recalling *immediately* after the three books *About the Soul* and before the books on life in general. The reason he gives is as follows:

Sed quia oportet per magis similia ad dissimilia transire, talis videtur esse rationabiliter horum librorum ordo, ut post librum *de Anima*, in quo de anima secundum se determinatur, immediate sequitur hic liber *de Sensu et Sensato*, quia ipsum sentire magis ad animam quam ad corpus pertinet: post quem ordinandus est liber de somno et vigilia, quae important ligamentum et solutionem sensus. Deinde sequuntur libri qui pertinent ad motivum, quod est magis propinquum sensitivo. Ultimo ordinantur libri qui pertinent ad communem considerationem vivi, quia ista consideratio maxime concernit corporis dispositionem. (*In Librum De Sensu et Sensato Commentarium*, Lectio I, n. 6)

Whoever has studied the fifth and sixth books of the *Elements*, the proportional books, will find the arithmetical theorems in books seven through nine very similar. And therefore they are reasonably ordered after them.

After the order in which the matter is considered, it is necessary to unfold *how* it is considered. Every reasoned out knowledge defines its matter and

demonstrates what belongs to it. Hence, it is necessary to see how things are defined and demonstrated in the *Elements*.

The geometer abstracts from sensible matter and, hence, from motion, but not from what is called understandable matter which is the imaginable matter of the continuous. And, in general, the geometer can see his beginnings in the imagination and it is to the imagination that his judgments return.

For the geometer, the cause is more known than the effect (contrary to what is found in most of our knowledge.) He reasons, for example, from the straightness of the intersecting lines to the effect of this straightness, which is the equality of the opposite angles formed by the intersecting straight lines. In this example, we also see the cause by which the geometer demonstrates, which is the formal cause. The demonstrations of pure mathematics excel in rigor and certitude those in other forms of reasoned out knowledge.

## THE AUTHOR OF THE ELEMENTS

Euclid probably received a mathematical education in Athens from the pupils of Plato. Later, Euclid taught in Alexandria and founded a school there. He flourished around 300 B. C. He was after Eudoxus and Theaetetus and before Archimedes.

Duane H. Berquist