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THE NOTION OF LIMIT

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INTRODUCTION

The present dissertation aims to show two things: first, that the method of limits is not confined to mathematics; secondly, that the method of limits may provide us with a key for a more fruitful reading of philosophical texts presenting a platonic outlook on things.

The work will be divided into three parts. The first part will be devoted to a mathematical illustration of the method of limits. The reason for this procedure is obvious enough. No one questions the usefulness of the method of limits in the field of mathematics and of the sciences subordinated to mathematics. Furthermore, while our intention is to show that this method far exceeds the field of mathematics, its application in mathematics is by far the most striking and clearest, so much so that to whatever other field it is applied, we always find it at least convenient to lean back on some mathematical analogy to sustain and clarify the application. Indeed, it might even be asked whether its application to

to other fields is but an extension of the mathematical theory, or whether the mathematical theory is itself but one instance of a method more general and logically prior.

The second part will be an analysis of the ideas underlying this method, mainly as exemplified in the mathematical theory of limits. The analysis will be made in aristotelian terms. But, in order not to arouse unduly either the historians or the mathematicians, we add that we do not thereby mean to imply that whatever we shall analyse or interpret in aristotelian terms is to be found in Aristotle or in the aristotelian tradition. The reader may reject the label; we are interested in the theory alone. The important matter will be what we use the label to signify in the theory, not in history. In this second part we shall present some non-mathematical illustrations, resting on classical aristotelian and thomistic texts.

In the third part, we shall present texts from Greek and Modern philosophy that are relevant to the method of limits. Our comments on them will be brief; it should be sufficient merely to read them in the light of what shall have preceded them.

In our handling of all these texts, ancient, mediaeval, and modern, we shall leave entirely undetermined what the intention of the particular author might have been. We shall be satisfied to show what we think <u>should</u> be intended if the text is to make sense, leaving it to the historian to devise, in his own strange way, justifications for attributing sense or nonsense to the author himself. It is a historical fact, however, beyond the reach of the historians, that this dissertation is the result of an attempt to understand these texts. In all justice, therefore, we are indebted to them at least as to an occasion.

Throughout the development of this work we shall be confronting certain highly embarrassing positions, which we are compelled to maintain in face of the quasi-universal rejection by what is called the "aristotelian and thomistic tradition". Again we must point out that our intention is not what is to-day called "historical", although we do intend to suggest a key to the reader of these and other such texts, that is to the reader whose purpose is to learn from them what is the truth of things. Let us consider a concrete example.

we shall try to establish that, according to the method of limits, the reason for the specific difference of things is not on the part of the form but on the part of the matter. But is not this a well-known platonic error, universally rejected by the whole aristotelian and thomistic tradition? Our answer will be that whatever Plato or the Platonists intended, the position as formulated is entirely correct within the bounds of the method of limits. If, then, the method of limits is legitimate, the unconditional rejection of the position has amounted to throwing out the baby with the bath. Hence, our purpose is, not to defend Plato, but to defend the position we formulate and which happens to be attributed to Plato, and rejected as platonic.

PART 1

MATHEMATICAL ILLUSTRATION OF THE METHOD OF LIMITS

Mathematical Illunstration of the Method of Limits

An attempt to understand the incalculable value of an irrational number (incalculable, that is, in terms of integers or common fractions), always brings to the fore several mathematical concepts essential to the right to the right understanding of the Method of Limits. Such terms as limit, itself, functions, variable, infinite progression, and others are illustrated in any careful answer to the apparently simple question: "What is the square root of 2?" The problem implied here is the very one familiar to the "Ancients" under the form of "incommensurable lengths." 1

In order to find the square root of 2, it would be necessary to find the number which, multiplied by itself, would exactly equal 2. Now, in the search for this number, it soon becomes evident that the number 1.4 is too small, while the number 1.5 is too large-- $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$. To determine the number, then, one could begin to increase 1.4 (with values that we shall here suppose to be arbitrarily chosen), or to decrease (in the same way) 1.5, keeping a continual watch on the progress, upward or downward, of the corresponding square-- e.g.:-

1.4------------1.96
1.41-----------1.9881
1.414----------1.999396
......----------......
1.41421356-----1.9999999732880736

1.41421357-----2.0000000145723449
......----------......
1.415----------2.00225
1.42-----------2.0146
1.5------------2.25

This progression of 1.4 is endless, as is the progression of the number dependent on it, 1.96. However, just as 1.96 is drawing ever nearer and nearer to a definite number, viz. 2, so 1.4 is drawing ever nearer and nearer to a number that is definite but inexpressible in terms of the familiar integers or ordinary fractions, viz. the irrational number $\sqrt{2}$. The important point here is that 1.96 will never reach 2, no matter how increasingly near it may draw, and 1.4 may take on millions of decimals but will never reach $\sqrt{2}$. All this holds, "servatis servandis," for 1.5 and its dependent 2.25. It is interesting to note that 1.4 and 1.5, despite their progress in the direction of each other, will never meet. At first, it seemed easy to locate the square root of 2 somewhere between these two numbers, separated from each other by a mere matter of .1. Later it became necessary to search for the square root of 2 somewhere between 1.41421356 and 1.41421357, the difference here

being only .00000001. Actually, the difference between these two apparently converging numbers, though constantly diminishing, will never be eradicated.

Another point to be noticed is that 1.96, as it continually increases, is related to 2 as it is to no other number. It is true that the nearer it approaches to 2, the nearer it necessarily approaches to 3 or 4; but that is only incidental, i.e. 1.96 approaches 3 or 4 only because any approach to 2 necessarily implies an approach to a higher number, and 1.96 is approaching 2. If, however, 1.96 were ever to attain 2, then there would be an end to this approaching and it would be seen that all along 3 and 4 were never intended, so to speak, in the approach. The approach to 2 is what is known as "tending to a limit".

VARIABLE

This simple example furnishes us with an illustration of dependent and independent variables. When we chose 1.4 as a square root to initiate our search for the square root of 2, we thereby determined the square that was to lead the way toward 2 itself, viz. 1.96. When we increased 1.4 to 1.41, 1.96 was necessarily and proportionately increased to 1.9881. Every change in the column of square roots caused a corresponding, dependent, and proportionate change in the column of squares. In other words the square root is a variable, receiving different values arbitrarily assigned (as we deliberately supposed), while the square is a variable, receiving values, not arbitrarily assigned, but necessarily determined by the continual dependence on the varying values of the square root. This, then, is called a dependent variable, while the former is an independent variable. This arrangement, of course, is not absolute and may be reversed. This is easily seen from the fact that we could just as easily have arbitrarily assigned the varying values to the square, thus making the square root the dependent variable. Usually, this distinction between variables (and between variables and constants) is illustrated in some algebraic xy formula, e.g.:- if in the slope-intercept form, $y = mx+b$, we let the constants m and b have the values 2 and 3 respectively, we have an equation with two undetermined variables, i.e. $y = 2x + 3$. If we choose to give definite values to x, we thereby make x the independent variable, and with every new value assigned to x there is determined a corresponding value of y, the dependent variable.

FUNCTION

Here we have the elements of the notion of "function". In the example just given, y is a function of x. In general,

> "when one quantity depends upon another in such a way that the first is determined when the second is specified, the first quantity is said to be a function of the other." 2

The idea of function finds a perfectly clear expression in the brief definition of Leathem:

> "The essential feature of a functional relation is simply a dependence of the value of the dependent variable (or function y) on the independent variable (x) by some mathematical rule or rules, formula or formulae, so that when a value is assigned to the independent variable, the value of the dependent is calculable by the rules or formulae without doubt or ambiguity."3

It is not at all necessary that the dependent variable be regarded as such absolutely. These qualities of the variables are interchangeable, since a functional relation is really a law of mutual dependence. In our original example, the square was a function of the square root, and the difference of the square root below $\sqrt{2}$ and the one above was a function of both of them.

Other examples of functions are such standard ones as the formula, $A = \pi r^2$, which is the mathematical formulation of the functional relation of the area of a circle to the radius. Here π is the absolute constant, r is the independent and A the dependent variable. In the falling-body formula, $d = \frac{1}{2} gt^2$, g (the acceleration due to gravity) is the constant, d (the distance traveled by the body) is the dependent variable, provided it depends on t for its value, and t (the interval of time) the independent. If t were to become the dependent variable and d the independent, the formula would then read: $t = \sqrt{\frac{2d}{g}}$. In either case, one is the function of the other. There are very many types of functional formulae, in fact

> "any combination of mathematical operations, whether algebraic, trigonometric, logarithmic, or an operation taken from the infinitesimal calculus or other branches of mathematics if applied to a variable x, will generally yield another variable which is entitled to be called a function of x." 4

Furthermore,

> "The idea of function, far from being confined to abstract mathematics, occurs frequently in science and even in ordinary experience. Thus, for a fixed electromotive force (E), the current (I) passing through a circuit is a function of the resistance (R), according to a simple formula known as Ohm's Law, I = E/R. If, by use of a standard cell, E is fixed say at 2 volts, and we put in succession various resistances into the circuit, the value of the current will be determined by the resistance used. If R is, for instance, 5 ohms, then I is 2/5 ampere. If we are free to change cells as well as resistances, that is, if E as well as R is variable, then I is a function of the two variables E and R. Functions of more than two variables also exist." 5 ← (?)

Conventionally, x is usually regarded as the independent, and y as the dependent variable; and the symbol f(x) is introduced to designate any function of x. Since y is the function of x, y = f(x).

this symbol, f(x), is, in the words of Leathem, "an abbreviation for some formula representing mathematical operations on the number x;" and y is the result of those operations on x, i.e. a certain value of which f(x) is merely the symbol or "abbreviation". Consequently, the formula, y = f(x), means merely that y is a function of x, without indicating what particular function is meant.

PROGRESSION
~~PROGRESSION~~

In the consideration of the problem of finding the square root of 2, there was an illustration of the mathematical notion of "progression" or "series". Perhaps it would be better to say that the notion of "series" was suggested by the sequence resulting from the unending increase of the number 1.4 and its function 1.96; for, strictly speaking, a series is a succession of terms which proceed according to some fixed law. A series is finite, if the number of terms is limited; infinite, if the number of terms is unlimited. The most common example is the series of odd numbers (1,3,5,7,...), which is infinite. A sequence in which each of the terms is derived from the preceding by adding to it a fixed amount (the "common difference") is called an "arithmetic series"; e.g. 3,6,9,12,15 is a finite (five terms) arithmetic series with 3 as the common difference. If the common difference were

to be added unendingly, the series would then be, of course, infinite. A sequence in which each of the terms is derived from the preceding by multiplying it by a fixed amount (the "common ratio"), is called a "geometric series"; e.g. 2,4,8,16,32 is a finite (five terms) geometric series with the common ratio, 2.

LIMIT Finally, the example of the square root of two introduces the notion of "limit". The numbers 1.4 and 1.5 tend, in an upward and a downward direction respectively, to the irrational number $\sqrt{2}$; and the numbers 1.96 and 2.25 tend, in an upward and a downward direction respectively, to the rational number 2. The tendency on the part of these four numbers is "endless" but not "limitless", i.e. there is a number which, if attained, would bring this tendency (the continual increase or decrease) to a close; but since this number, no matter how closely approximated, is never attained, the tendency toward it never halts. This number, which marks the goal, the unattainable terminus of a tendency, is called a "limit". It is this notion of limit that may now be discussed more easily, after the brief considerations of the notions of variable, function, and series.

In general a limit may be defined as a fixed value or form which a varying value or form may approach indefinitely but cannot reach. The more or less standard definition of limit is the following:-

> "When a variable x approaches a constant c in such a way that the difference between them in absolute value becomes and remains less than any preassigned quantity, however small, x is said to approach c as a limit."

or:-

> "If a variable x approaches more and more closely a constant value a, so that a - x (i.e. according to numerical or absolute value) eventually becomes and remains less than any preassigned number, however small, the constant a is the limit of x."

or:- "A function f(x) has the limit L at a value a of its argument x, when in the neighborhood of a its values approximate to L within every standard of approximation."[8]

The definition implies two theorems that may be expressly stated as follows:- 1) If a variable never decreases and never becomes greater than a fixed number, then it approaches a limit which is not greater than the number. 2) If a variable never increases and never becomes less than a fixed number, the variable approaches a limit which is not less than the number. Sometimes the limit of a variable is zero, in which case the variable is called an infinitesimal. Sometimes the limit of a variable is infinity, in which case the variable is called an infinite.[7] This literal contradiction (limit is infinity) merely means that the variable progresses "with such a general trend of in-

crease of numerical value as will finally transcend any number set up as a barrier in its path."

This general idea of limit may be clarified in its first and customary illustration. The fraction $\frac{n+1}{n}$ is a function of the independent variable n, to which we are going to assign successively increasing values, beginning with some positive integer. For the values of n, we may assign an arithmetic series, with 10 for the first term and 1 for the arithmetic mean. As n progresses through the values 10,11,12,13 ......, its dependent variable, $\frac{n+1}{n}$, progresses accordingly:- 11/10, 12/11, 13/12, 14/13.......101/100, 102/101, 103/102, 104/103.......100001/100000, 100002/100001, 100003/100002, 100004/100003, etc. Each term is less than the preceding, but despite this continual decrease in value, the decrease is not unlimited. Here, as before, there is a difficulty of expression. This series of the values of the fraction is endless, yet in this endless procession there is a limit. Every term is greater than zero, greater than ½, greater, in fact, than 1. Unity is the greatest number of which it can be said that every term of the series is greater than it. At no stage in the unending progress downward will the fraction $\frac{n+1}{n}$ be less than the number 1. It may pass $1\frac{1}{10}$, or $1\frac{1}{100}$, or $1\frac{1}{1000}$, but it will never pass 1. If e represents any small positive number, as small as desired, $\frac{n+1}{n}$

will pass 1 + e in its tending to 1. In other words, given any small number, no matter how small, there will always be a stage in the progress of this fraction beyond which the difference between the fraction and the number 1 will always be less than this small number. While there is no number greater than 1 that $\frac{n+1}{n}$ will not pass, nevertheless on the other hand, $\frac{n+1}{n}$ will never reach 1 itself. In this case, the number 1 is the limit of the variable $\frac{n+1}{n}$, the number of all numbers of which it can be said:- "this number cannot be passed, yet it is impossible to name any greater number, however near, which is not passed in the downward progress of $\frac{n+1}{n}$ due to continual increase of n." Since we assigned as values of n only positive integers beginning with 10, there is no value of n for which 1/n=0, and therefore no value of n for which $\frac{n+1}{n}$ = 1. That is just the point, $\frac{n+1}{n}$ will never equal 1; its <u>limit</u> equals one, and <u>it</u> tends to its limit as to a goal but never attains it. Its value may lie between 1 and 1 plus a very small number continually getting smaller, but this value, not being a limit, will always be passed. In the words of Leathem,

> "A limit, then, is the goal of an endless progress of a variable, a number to which approximation is ever closer and closer. And the test of approximation to a limit is the impossibility of setting up, between the limit and the approaching variable, a barrier number which may not be ultimately passed by the variable in its progress."[8]

Another point, to be notice here, already seen before, is the difference between mere "approaching" and "tending to a limit". It may be said that as <u>x</u> tends to 3 as a limit, it is also approaching (in the sense of "mere approaching") to 4. The variable <u>x</u> will never reach 4, because between it and 4 there is an "effectual barrier," viz. 3. Neither will it ever attain to its limit, 3; and this, not because there is any "effectual barrier" between it and 3, but because there is an infinity to be traversed. At a point in this "traversing", the difference between <u>x</u> and 3 will become and remain smaller than any arbitrarily chosen number, however small; but there will always be some difference between <u>x</u> and 3.

The example given above was an illustration of one of the theorems mentioned previously, viz. if a variable never increases and never becomes less than a fixed number, then it approaches a limit which is not greater than the number. If a variable assumes the sequence of values $2, 3, 3\frac{1}{2}, 3\frac{3}{4}, 3\frac{7}{8}, 3\frac{15}{16}, 3\frac{31}{32}, 3\frac{63}{64}, \ldots\ldots 4$, it never exceeds 4; i.e. it approaches a limit which is not greater than 4 (as a matter of fact the limit is exactly 4). This is an illustration of the other theorem, viz. if a variable never decreases and never becomes greater than a fixed number, then it approaches a limit which is not greater than the number.

An important point to be noticed in the example of the fraction, $\frac{n+1}{n}$, is that it is a function, and that its limit is, properly speaking, the limit of a function. This first fact could be expressed in the formula, $\underline{n+1}/n = f(n)$. However, not only does the value of this fraction depend on the value of the independent variable, but its tending in general and its limit depend on the tending and limit of the independent variable. The limit of the fraction in this case is equal to 1, as the independent variable tends to infinity as a limit (which means, literally, that the independent variable has no limit).

The limiting value of a function requires separate consideration. Any endless progress of $\underline{x}$ generally determines a corresponding endless progress of $f(x)$. If there is related to this progress a number $\underline{c}$ such that, if we select any small positive number $\underline{s}$, no matter how small, there is always a corresponding definite stage in the progress of $f(x)$, after which it is always the case that $|f(x)-c| < s$; then, $f(x)$ in this progress tends to $\underline{c}$ as a limit. As a matter of fact, a function of a variable may <u>or may not</u> approach a limit as the variable itself approaches a limit. In some cases, even, the function may not be defined, i.e. may lose all meaning or value when the independent variable takes certain values. This is clear from the formula:-

$$f(x) = \frac{x^2-1}{x-1} .$$

Here, the fraction (which is the function of x) has meaning or value for every value given to x, with one exception. If x equals 1, then $f(x) = \frac{0}{0}$, which is meaningless. In such a case, the function of x (which may be expressed as f(1) in this particular instance) is said to be undefined. The limit of a function does not exist, or is said not to exist, when the function is actually approaching two different limits because the variable is approaching its limit both through values larger than that limit and through values smaller than it.

If, however, the variable tends to a definite limit, and its function also tends, in consequence, to a limit, the fact is expressed in the following formula:

$$\lim_{x \to a} f(x) = 1$$

This means that the limit of $f(x)$, as x tends to or approaches a, is 1.

In this matter of the limit of a function, there are four cases to be considered:-

1) The function and the limit exist and are defined.

This case is illustrated in the function $f(x) = x^2 + \frac{1}{x}$ and the limit $\lim_{x \to a} f(x)$, where $a = 2$. Since $\lim_{x \to 2} x = 2$,

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2 + \frac{1}{x}) = \lim_{x \to 2} (x^2) + \lim_{x \to 2} (\frac{1}{x}) =$

$(\lim_{x \to 2} x)^2 + \frac{1}{2} = 4\frac{1}{2}.$

The function itself is defined, for no value of x (we are adhering to positive integers) invalidates the formula x and the limit exists (i.e. is neither infinite nor double).

2) The limit exists, but the function is not defined at a certain point.

This case is illustrated in the example given above, viz. $f(x) = \frac{x^2-1}{x-1}$, and $\lim f(x)$ where $a = 1$. The function of the independent variable x is defined for every value given to x, with one exception. When $x = 1$, the function $f(1)$ $\left(\frac{1-1}{1-1}\right)$ is not defined, because $\frac{0}{0}$ is meaningless. However, the limit of the function exists:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2$$

3) The function is defined, but the limit does not exist.

This case is illustrated in the example

$$f(x) = (x+1) \quad (x-3)(x+8)$$

Here, f(x) is defined for any value of x; its value is a real number if x equals or is greater than 2, an imaginary number for all values of x less than 2-- except for the one value x -1. (In either case strictly speaking, the function is defined). However, if we limit ourselves to the domain of real numbers, f(x) cannot be regarded as approaching a limit as x approaches -1.

4) The limit does not exist and the function is not defined.

In other words, $\lim_{x\to a} f(x)$ does not exist and f(a) is not defined. This is clear from the example $f(x) = x+\frac{1}{x}$, when $a = 0$. Substituting the value of a for x, f(x) becomes f(0) and impossible because of the $\frac{1}{0}$. Therefore, f(x) as f(a) is undefined. If x approaches 0 (that is, tends to it without ever reaching it) then $\frac{1}{0}$ increases in numerical value indefinitely, i.e. without limit. Therefore, $\lim_{x\to a} f(x)$ does not exist.

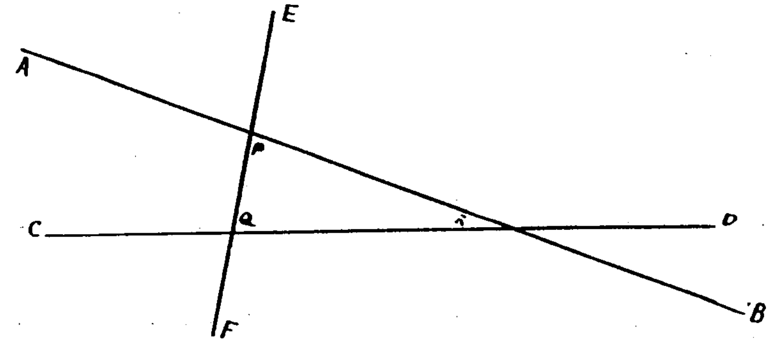
There is a sense in which all geometrical figures may be considered as limits. If a point is that which has position, but no magnitude, and a line is that which has length without breadth, then all these visible, sensible points and lines appearing in geometrical diagrams are tending as to a limit to these non-sensible points and lines which are of the true subject of geometry. They are variables, as it were, which the geometrician uses in his discussion of their limits. That is why, as Aristotle pointed out, that the geometrician does not utter falsehood in stating that the line which he draws is a foot long or straight, when it is actually neither. Aristotle said that the geometrician, in fact, does not draw any conclusion from the being of the particular line of which he speaks, but from what his diagrams symbolize. What those diagrams symbolize may be regarded as their limit, in an analogous sense. A stricter exemplification of limit in the realm of geometry may be found in the relations between parallel lines, curved and straight lines, polygons and circles.

According to Euclid, parallel lines are such as, being in the same plane, do not meet however far they are produced in both directions. The discussion here of parallel lines as an illustration of the idea of limit involves the Euclidean Parallel Postulate, which is here repeated:

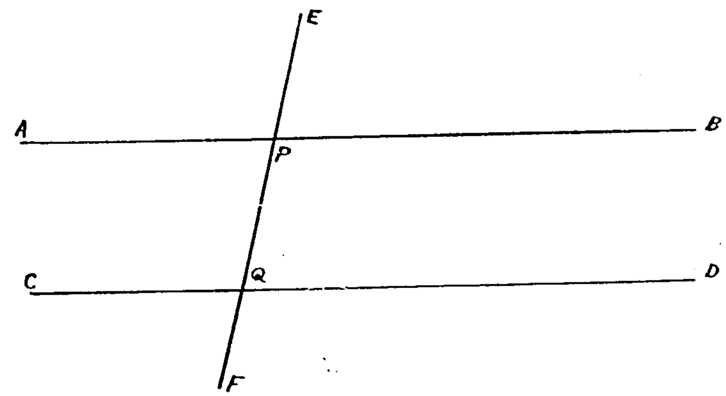
> "If a straight line cut two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, will at length meet on that side on which are the angles which are together less than two right angles."

However, a set of parallel lines may be considered as the limit of a set of intersecting lines. This can be seen in the following example: The two straight lines AB and CD (cf.Fig.1) are cut by the straight line EF in such a way that the two interior angles on the same side of EF are together less than two right angles. The lines AB and CD are intersecting lines, and together with EF form the triangle PQR.

In the triangle PQR, the corners P and Q are fixed while the point at R is a variable moving fur-



(Fig. 1)

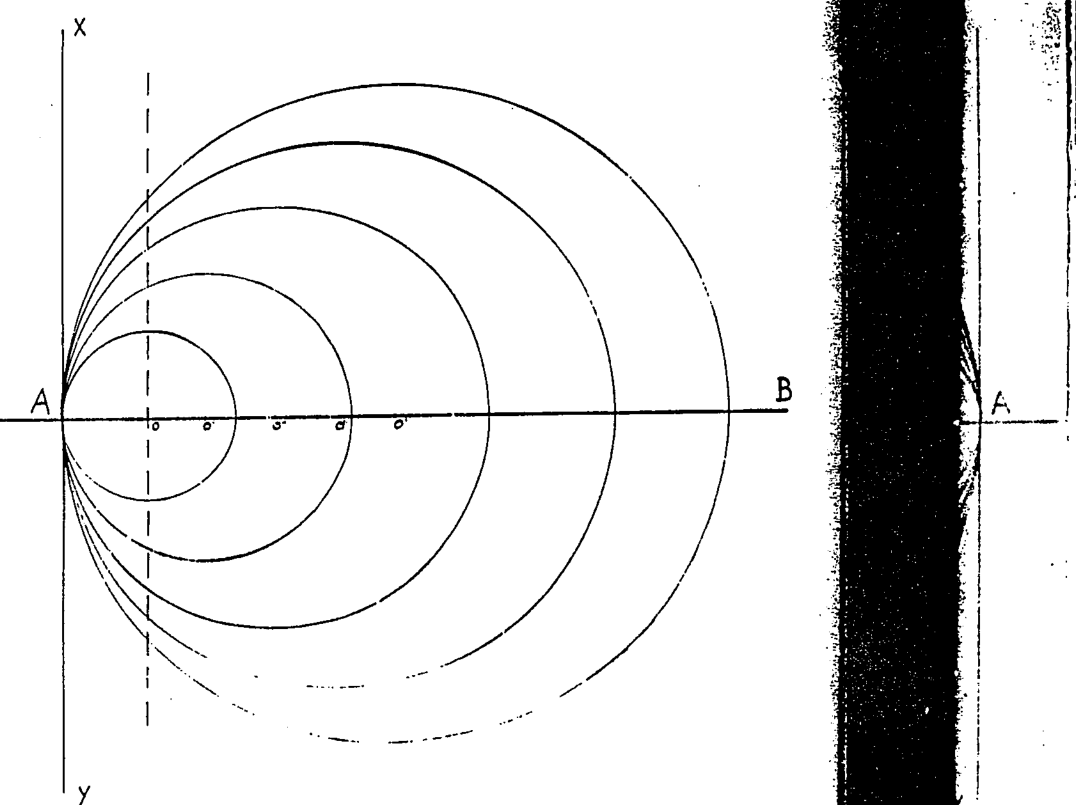


(Fig.2)

ther and further along the line CD. In fact, R may be regarded as tending to infinity, or the line QR passing D and extending indefinitely. The sum of the angles at P and Q (which is less than two right angles) is a function of the variable point at R (or QR). The further away R moves from Q, the smaller the angle at R becomes, and the greater the angle at P becomes. The further the line QR extends, the closer to the value of two right angles does the sum of the angles at P and Q approach. If the sum of P and Q were to equal two right angles, then the lines AB and CD would be parallel. Therefore, if (as the point R tends to a position ever further and further away from the point Q) the sum of the angles at P and Q tend to the value of two right angles as to a limit, then the intersecting lines AB and CD tend to the configuration of two parallel lines cutting EF at P and Q respectively. See Figure E for the limit configuration. The second figure, the limit configuration, is essentially different from the progressively changing configuration of which it is the limit. Furthermore, the limit will never be attained so long as there is a point R, where AB cuts CD. This is quite obvious. Since, however, the line CD may be of infinite length (i.e. extend indefinitely), there will always be an R, which may move without limit (i.e. tend to infinity) along the infinite line CD.

Here precisely is the notion essential and fundamental to the idea of mathematical limit. The movement of the point R away from the point Q (and the point R implies always the intersection of the two lines AB and CD) is not so unrestricted that it may disappear altogether. Despite this restriction, it is unrestricted in the sense that there is no end to the movement of the point. There is here an infinity within limits, and it is this "restricted infinity", so to speak, that is responsible here (as it was in the case of the "arithmetical area" between any two consecutive integral numbers) for the very notion of mathematical limit.

A simple and clear geometrical illustration of the theory of limits may be had in the case of curved and straight lines. It can be shown that a straight line is the limit of a curved line, or that a curved line is the limit of a straight line. In Figure 3, for endlessly increasing radius of the circle, part of the circle (on the left hand side of the dotted line) tends to a straight line as to a limit, namely the straight line XY drawn through a perpendicular to AB. This case of limit may be proved in the following way. On the straight line AB (see Fig.4), let A be regarded as a fixed point and O as a moving point. With O as center and OA as



(Fig. 3)

as radius describe a circle corresponding to O's first position. Draw a straight line, AN, through A, perpendicular to AB. Draw a line through N, parallel to AB and meeting the circle at the points P and P'. Then from the point P let fall the line PL, perpendicular to AB, and let A' be the opposite end of the diameter through A.

Since AL : LP :: LP : LA' (Euclid; Bk.VI, Prop.13)

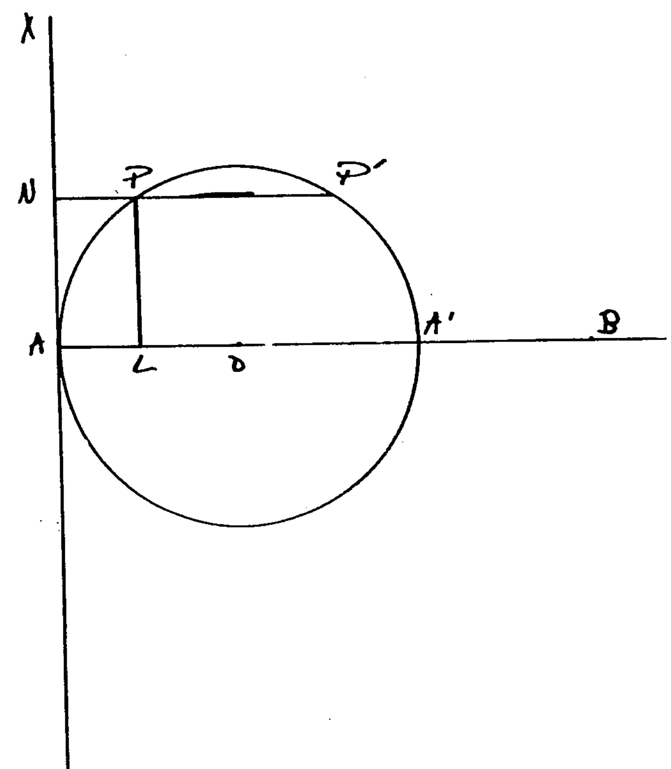
then $AL \cdot LA' = LP^2$

and $NP \cdot LA' = AN^2$ (by substitution of equivalents)

$\therefore NP = \frac{AN^2}{LA'}$

If O is moved endlessly further from A, then LA' increases endlessly. Since LA' is the denominator of the fraction to which NP is equal, it follows that with the endless increase of that denominator the fraction itself decreases endlessly. Therefore NP, the distance between the circular arc and the straight line decreases endlessly. In other words, part of the circle which is always circular and never a straight line, tends to the form of the straight line as to a limit.

The exact opposite of this case of limit is the tendency of a straight line to the form of a circular arc as to a limit. The familiar example of the polygon inscribed in a circle or circumscribing a circle, with the number of its sides increas-



(Fig. 4)

ing endlessly, furnishes an illustration of the case at hand. It will be sufficient here merely to produce the figures (see cf. Fig.5 & 6).

There are a number of different ways of regarding the illustration of the inscribed polygon (fig.5). One way is to regard the arc of the circle as the limit to which the side of a triangle tends through successive and unending multiplications. Another way is to consider the entire circumference as the limit of the perimeter of the inscribed polygon of n sides. Letting each side have the length s, then since the number of sides is increasing endlessly, π may be defined as a limit itself:

$$\pi = \lim_{n \to \infty} \frac{ns}{d}$$

This means that π is the limit of $\frac{ns}{d}$ as n increases without limit or "tends to infinity".

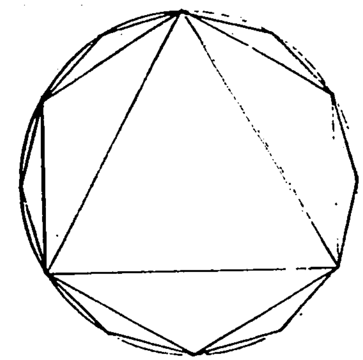
In Trigonometry the theory of limits finds its most common illustration in the limit of the ratio of a vanishing angle to its sine. This may be expressed in the following formula:

$$\frac{x}{\sin x} \longrightarrow 1 \text{ when } x \longrightarrow 0,$$

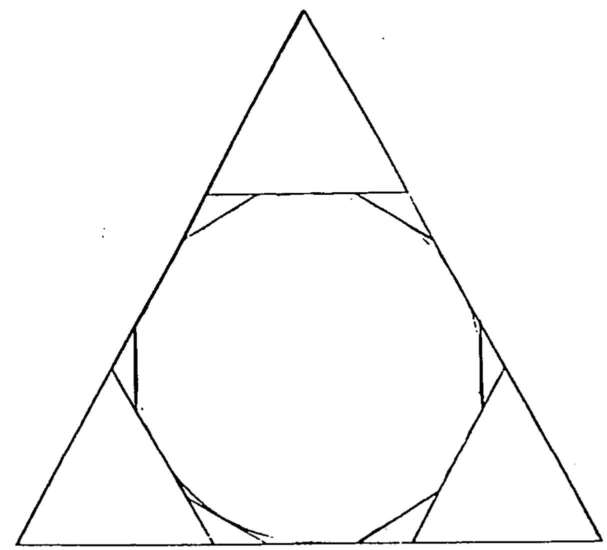
or

$$\lim_{x \to 0} \frac{x}{\sin x} = 1.$$

Here, the ratio of the angle to its sine is a function of the angle itself, which is a variable tend-



(Fig. 5)



(Fig. 6)

ing to zero. If x were to be regarded as having attained its limit (0), the function would have no meaning; but the problem is concerned with the function when x tends to zero, not when it equals zero.

Let the arms of an acute angle (see Fig.7) whose vertex is O and circular measure 2x cut a circle of center O and radius r in P and Q. Let the tangents at P and Q meet in T, and let OT intersect PQ in N and the circle in A. Then PQ is perpendicular to OA, and the angle AOP is x.

$$\sin x = \frac{PN}{r};\ \text{mes } x \text{ (in radian)} = \frac{\widehat{PA}}{r};\ \tan x = \frac{PT}{r}$$

The arc PAQ lies inside the triangle PTQ and is everywhere convex towards PT and TQ. Hence, since the shortest path from a given point to a straight line (whether polygonal or curved paths be considered) is that along the perpendicular,

$PN + NQ < \widehat{PA} + \widehat{AQ} < PT + TQ$ $PN + NQ < \widehat{PA} + \widehat{AQ} < PT + TQ$

$2PN < 2\widehat{PA} < 2PT$ $2PN < 2\widehat{PA} < 2PT$

$PN/r < \widehat{PA}/r < PT/r$ $PN/r < \widehat{PA}/r < PT/r$

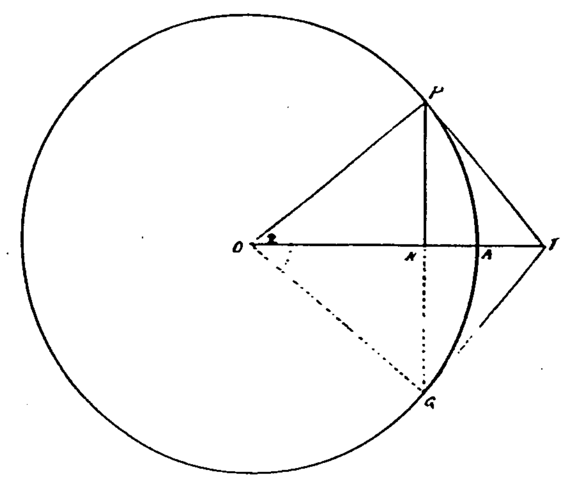
$$\sin x < \text{mes } x \text{(in radian)} < \tan x$$

$$\therefore\ \tan x > x > \sin x$$

Division by sin x gives $1/\cos x > x/\sin x > 1$

$$\lim_{x \to 0} 1/\cos x > \lim_{x \to 0} x/\sin x > \lim_{x \to 0} 1$$

$$1 > \lim_{x \to 0} x/\sin x \geq 1.$$



(Fig. 7)

A final point to be considered, briefly, is the use of the infinitesimal in the differential and integral Calculus;[9] for the infinitesimal is itself a variable tending to a limit, viz. zero, and the limit idea is well illustrated in those calculations in which the infinitesimal plays a part.

When y is a function of x ($y = f(x)$), a change in the value of x, called an <u>increment</u> of x, will, generally, give rise to an increment of the function. It is both interesting and profitable to compare the corresponding increments of the function and its independent variable. The increment of the independent variable is usually expressed by the symbol Δx; and that of the function, by Δy. Since y changes to the extent of Δy for every corresponding change of Δx on the part of x, $\frac{\Delta y}{\Delta x}$ represents the average change of y per unit change of x, or the average rate of change of y with respect to x in the interval Δx. It may happen that the increments of y do not maintain a uniform rate. In order to obtain the rate of change of y with respect to x for the initial value of x, the method of limits must be introduced. As x approaches zero, $\frac{\Delta y}{\Delta x}$ may or may not approach a limit. If it does, the limit in question is the rate of change that is being sought. This limit is called the <u>derivative</u> of y with respect to x, and may be represented by the symbol $\frac{dy}{dx}$. We have then

$$\frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}.$$

The procedure involved in arriving at this conclusion is as follows:-

We start from the relation

$$y = f(x)$$

and give x the increment Δx; whereupon y becomes

$$y + \Delta y = f(x + \Delta x)$$

Subtracting y from $y + \Delta y$, we obtain

$$\Delta y = f(x+\Delta x) - f(x)$$

Next we divide Δy by Δx; the quotient,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is the average rate of change of y with respect to x in the interval from x to $x+\Delta x$. Finally, passing to the limit as Δx approaches zero, we obtain the derivative of y with respect to x:

This result represents the rate of change of y with respect to x at the beginning of the interval.

This idea of the derivative may be represented geometrically. (cf fig. 8)

Let the curve represent the graph $y = f(x)$. Letting x and $x + \Delta x$ be the initial and final values of the independent variable, take two points

on the curve corresponding to these values, viz.
P (x,y) and Q $(x+\Delta x, y+\Delta y)$.

RQ/PR is the slope of the secant line through P and Q; but since (as is evident) $\Delta x = PR$ and $\Delta y = RQ$, then

$\frac{\Delta y}{\Delta x}$ = slope of secant line through $P(x,y)$ and $Q(x+\Delta x, y+\Delta y)$

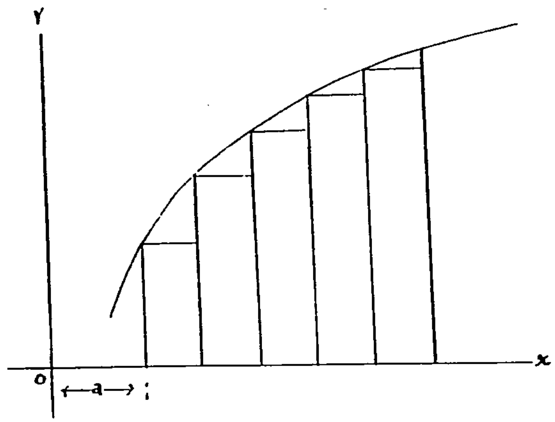
If Q moves along the curve through such positions as Q', the secant line then rotates about P, its slope always being given by $\frac{\Delta y}{\Delta x}$. As Q approaches P, the position of the secant approaches that of the line PT, the tangent to the curve at P. Therefore the slope of the secant approaches the slope of the tangent as a limit. Now, as Q approaches P, x approaches zero, therefore

$$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \text{slope of tangent at } P(x,y).$$

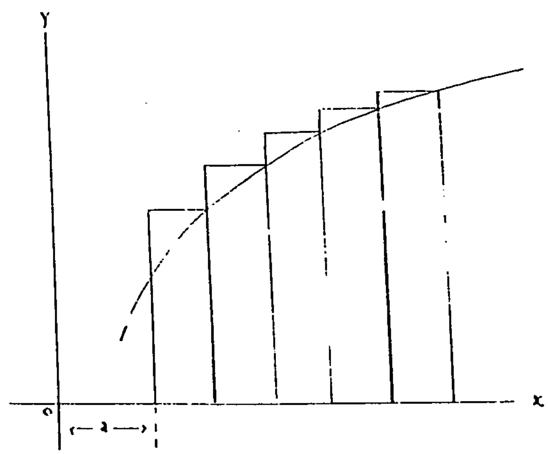
Therefore, we may say that the derivative of a function f(x), for a given value of x, is equal to the slope of the curve $y = f(x)$ at the point having as abscissa the given value of x.

As regards the derivative $\frac{dy}{dx}$, we may say that $dx = \Delta x$, i.e. the differential of the independent variable is the same as its increment. However, dy is not equal to Δy but to $\frac{dy}{dx}$ x, i. e. the differential of the function $y = f(x)$ is its derivative multiplied by the differential of the independent variable x.

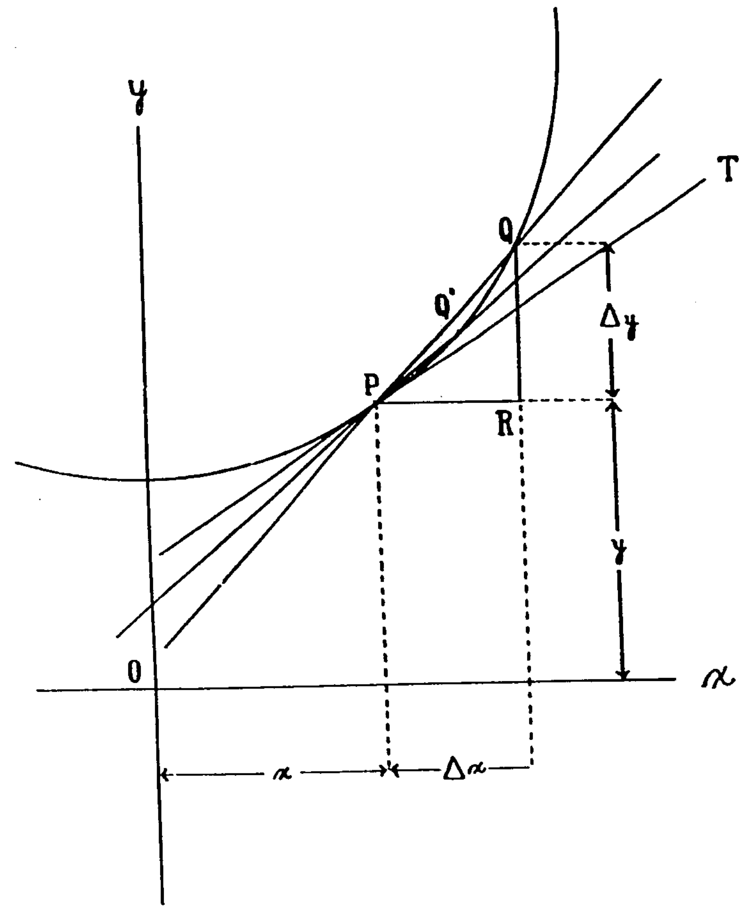
Finally, and in a few words, an area or any quantity which may be interpreted as an area, is readily expressed as the limit of a sum. The area, for example, under a continuous curve between two fixed ordinates may be approximated to any desired degree of accuracy by summing the areas of a sufficiently large number, n, of rectangular elements, either inscribed or circumscribed; and the exact area under the curve is the limit of this sum as n becomes infinite. (See Figs. 9 and 10).



(Fig.9)



(Fig.10)



(Fig.8)

PART II

ANALYSIS OF THE IDEAS UNDERLYING THE METHOD OF LIMITS

Analysis of the Ideas Underlying the Method of Limits

The following analysis of the notion of limit would not interest the mathematician. With respect to the subject of this analysis, the mathematician takes an entirely pragmatic attitude. The mathematical theory of limits works. This is an incontestable fact, like digestion. But even though all digestion were good, or bad digestion irremediable, some people would still like to know why it is that it can work. That is one of the differences between being like a man and being like a horse. Let us then step right into the yawning abyss of philosophy.

Descartes found great fault with Aristotle's analysis of movement. How could a man render so obscure what was apparently so simple? Indeed the simplicity is perhaps only apparent. As Professor Muirhead said: "....It may be well to remind (the mathematicians) from the side of philosophy that here, as elsewhere, apparent simplicity may conceal a complexity which it is the business of somebody, whether philosopher or mathematician............ to unravel."[10] Especially so when history shows that mathematicians have concocted some extremely foolish notions on the nature of the infinitesimal, notions which might have been avoided had the mathematicians shown more regard for non-mathematical analysis, or, we should add, had the philosophers themselves shown more truly philosophical interest in that kind of analysis. As A. E. Taylor has well put it:

> "Berkeley and others... condemned the Calculus for employing the notion of 'vanishing' magnitudes. They were quite right in saying that the theory of the Calculus, as formulated by its exponents, introduced vanishing magnitudes which are treated as somethings which are turning into nothings, and that to talk of such nothing-somethings is to talk nonsense. But the criticism really hit not the Calculus itself but only the inaccurate analysis its exponents had given of their own methods."[11]

We shall use the most elementary mathematical examples. As we shall see in a later chapter, the analysis itself will apply to non-mathematical entities as well. For the present it would be as well not to think of the latter.

1. Limit is a relative term. Nothing is limit in itself or with respect to itself. Limit is the limit of something. The very notion of limit implies two distinct terms, one of which is a variable and the other a constant. The constant is the limit to which the variable is ordered by the successive and uninterrupted assumption of values, in the direction of the constant.

Not all movement from one term to another is a tendency toward a limit as we here understand limit. In the first place, the two terms must be formally, specifically distinct: they must differ by definition, such as polygon and circle. The tendency toward the limit must be a tendency toward the difference as such. A limit which can actually be reached would be, in that respect, of the order of the variable and hence homogeneous with any possible value of the variable.

By this it is clear that the notion of limit as here taken is a technical one, not to be confused with the common notion of limit which may be defined as the ultimate value of a variable homogeneous with any one of the preceding values. This does not mean, however, that we rule out the common notion of limit. On the contrary, we not only suppose it as a prerequisite, but, as we shall show, we actually use it in the tendency toward a limit in the technical sense, for it is part of the very notion of the limit taken in the technical sense. (From now on we shall understand the single word "limit" as referring to the technical sense; if we wish to speak of the common notion, we shall indicate that explicitly). In fact, this is necessary, for if we did not, in some respect, consider limit as homogeneous with the values of the variable, the tending to limit would be meaningless. It might be said, in another respect, that limit is possible only where there is no limit (i.e. in the common or non-technical sense). Again we might say that in tending toward limit we tend to identify it with a common limit, so that, if limit could actually be reached it would be identical with a common limit.

The Variable Ordered to the Limit

2. Let us now consider more closely the ordered term or variable. Any variable implies both unity and multiplicity, identity and otherness, form and matter.

The unity of a variable is expressed by its definition; the multiplicity, by the different values it may assume. The identity of the variable means that its definition is identical for any one of its values; by its otherness, we mean the diversity contained within the variable on the part of its values, and not its otherness with respect to some other term. By the form of a variable we mean that which is expressed by the definition and predicable of anyone of its values, and not the form which is part of the definition itself. Likewise, by the matter of a variable we do not mean the matter which is part of the definition, such as genus, but the values of which the definition may be predicated, such as for polygon: triangle, hexagon, etc., which we may call the class of polygon. Any polygon is in the class polygon because we may apply to it the definition of polygon. The form of polygon is invariable; the variability of polygon is on the part of that of which polygon may be predicated, that is on the part of its matter. Hence, in this respect, the differences of polygon are due to the subject.

Hence, the notion of variable as we here take it embraces both the "metaphysical whole",(such as genus and difference) and the "logical whole", that is the subjective parts of which the definition may be predicated; the two wholes being related as form and matter. We make this point because there is another respect in which variable may be considered as a matter determinable by different values, just as genus is determined by a difference. Consider, for example, triangle, in relation to which polygon is as the matter, and three straight sides the difference; yet, triangle in turn becomes matter in view of scalene, etc.. Why we are interested in the first point of view which is logical, shall be clear from what follows.

5. We have called the matter of a variable the class of the variable. Now there is a respect in which all classes are closed:[15] they are restricted to the members which have the same definition. In this respect, the class of integers or the class of polygons is closed. This limitation follows from the identity of the definition of any member of the class, which opposes the class to any other class.

Thus, even if there were a class comprising an actually infinite multitude of members, it would still be finite in the respect under consideration.

That limitation of a class which is due to the identity of its form does not imply a limitation on the part of the multitude or variety of its members. The class constituted by the proximate species of triangle, that is: equilateral, isosceles and scalene, is a closed or finite class with respect both to the form and to the matter. The class of isosceles, however, or the class of scalene triangles is open; their possible varieties are indefinite within the limits established by the definition. We may therefore say of an open class that it cannot actually contain all its members.

This property of an open class may seem contradictory, since "all the members of a class" means "all the members contained within the class". However this contradiction appears only when we overlook the term "actually". An open class contains all its members "potentially", that is, there is no end to the multitude or the variety of members it can contain. We may therefore define an open class as a class "cujus est semper aliquid extra". An open class may then be called an infinite class, provided we take the term infinite for "cujus est semper aliquid extra". For, if there were a class with an actually infinite multitude of members, it

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would be a closed class, to which we could apply the definition of a perfect whole: "extra quod nihil est."

It is important to bring out this distinction between the respect in which every class is a perfect whole, and that in which some classes are essentially imperfect wholes. If we did not make this distinction, we might refer the "openness" of a class merely to our inability to reach the class in its perfect totality. In other words, we might suppose that an open class is fundamentally a closed class, that is, that in itself it has an actually infinite multitude of members, but that we, for some reason or other, cannot actually reach the actually infinite multitude. This, as we shall later show, would destroy the method of limits at its very root.

4. The class of the variable ordered to a limit must be an open or infinite class (we shall henceforth take infinite for the potential infinite). If the class were closed, its limit would be a common limit and nowhere. Thus infinity is essential to the variable implied in the notion of limit. But infinity is not the ultimate property required of the variable ordered to a limit.

We have already stated that the values of the variable must be ordered to the limit, as in the series 1, 1½, 1¾, ... 2. The order in virtue of which the variable is ordered, is on the part of the va-

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lues, not on the part of the definition itself, except, as we shall see later, by consequence. Now there are two kinds of order: accidental order, and formal or "per se" order. The order of individuals of the same species, considered in their pure homogeneity, is merely accidental, such as 1, 1, 1, 1,... But the order among various species is essential, for they differ formally, such as 1, 2, 3,.... [13] The values of the variable ordered to a limit must constitute a formal order.

We may now state more clearly that the formal variability of the variable which is on the part of the matter, is due to the proximate forms of the matter, as distinguished from the form of the variable with respect to which the proximate forms and their matter are as matter. Hence, when we say that the formal differences of polygon are due, not to the form polygon, but to the matter, we refer to the forms which are on the part of the matter, and not to that matter which is the proximate subject of these forms. Furthermore, this matter could account only for homogeneous otherness on the part of the matter of the variable. Thus, the difference between man and brute does not come from animal but from that of which animal is predicated i.e. from that with respect to which animal is form. We shall show later why we confine ourselves to this logical point of view.

Triangle is a variable whose proximate matter is equilateral, isosceles and scalene. The order of these species is formal. But it does not meet the requirements of a variable ordered to a limit. The order of this variable must have infinity; it must be both formal and open, as the series 1, $1\frac{1}{2}$, 1^3,....

The Limit of the Variable

5. As regards the limit, or fixed term or constant, much has already been said of it by implication. We may now state more explicitely that the limit may be compared to the variable in two ways: either to the form of the variable, or to its matter. In the first respect, variable and limit are absolutely heterogeneous; they differ by definition, and hence they are purely and simply irreducible. Therefore,

(a) as variable can have a limit by virtue of its form as such.

From this we may immediately conclude that if a variable could actually reach its limit, the form of the variable and the form of the limit would be both formally different and formally identical as to their proper definition, which is a contradiction.

However, when we compare the limit with the matter of the variable, we encounter a unity entirely foreign to the first comparison. For, any value of the variable is less different from

the limit than any preceding value. Hexagon is closer to circle than triangle. As the series goes on, the difference decreases. And since the series is essentially open, there is no limit to the decrease of difference. This comparability is therefore bound to the "more" and the "less" on the part of the matter of the variable, and they are a property of its formal order.

6. "More" and "less" are relative terms. Nothing is absolutely so. For example, with respect to their sides, hexagon is a greater polygon than triangle, and smaller than duodecagon. However, and this is to bring out the peculiar comparability we are here speaking of, if we can say that one polygon is greater than the other, meaning that it has more sides than the other, we cannot say that one polygon may be more polygon than the other; for, this would mean that what they have in common, that which is identical, is different. We say that they are more or less such or such. It is with respect to some form other than their proximate form, that is with respect to some form other than the proper form of the terms compared, and other than their common form— therefore, with respect to some other form which cannot be directly predicated of them. No polygon has the form of circle, neither as proper nor as common. But we do say that one polygon is "more like" a circle, or closer to it, than some other.

that we have so far stated is by no means proper to the matter of the variable ordered to a limit. We may say that isosceles triangle is more like equilateral than is scalene. Here the greater likeness is said with respect to another species of triangle, a species within the same series. Hexagon is more like duodecagon than is triangle, and such a comparison may go on indefinitely without reference to anything outside the series. The first principle of the order lies definitely within the series whose proximate common predicable is identical.

"The more and the less" characteristic of the variable ordered to a limit pervades the whole series within the limits of the proximate common form of the members, with respect to a form which lies beyond the series and which is the first principle of this peculiar comparability. Circle is the principle of the order of the variable polygon to its limit. That to which a thing is ordered is the principle of the order. And this should be noted, for, although triangle for example, is the principle of the series of polygons, it is only materially so with respect to the circle series as ordered to circle, a series of which triangle itself is a part. Again, it is because the circle series is [illegible] that we may say that the variable is ordered to the limit.

V. Another point to be explicitely stated is that there is no limit to the greater likeness to which the variable may approach. Thus, if we take some closed series, closed either essentially (such as the series of the proximate species of triangle) or closed by choice (such as the series of polygons from triangle to duodecagon), then, some determinate member of the series is the closest possible, or the most like the term to which it is said to be "more like". The "more like" is then the "most like", the "most like" being that which approaches most to likeness pure and simple. On the contrary, the "most like" of an open series ordered to a limit would be absolutely like to the limit, equal to it, or identical with it. The polygon most like to the circle would be a circle; the sum of $1 + \frac{1}{2} + \frac{1}{4} \ldots$ most like to 2 would be 2. The circle would be identical with the greatest possible polygon, and 2 identical with the greatest possible sum of $1 + \frac{1}{2} + \ldots\ldots$

This in turn brings out the importance of infinity in the notion of limit. It now remains to show that this infinity is meaningless without movement, so that movement is essential to the notion of limit; nothing is limit but with respect to some movement.

The Tendency toward the Limit

8. We cannot say that circle is the limit of regular polygon absolutely. Nor can we say that it is the limit of any given polygon, no matter how great, or of any given series of polygons, no matter how great the given series may be. If there were a greatest possible polygon, circle would in no sense be the limit of polygon. That is why we say that "circle is the limit of a regular inscribed polygon whose sides increase in number." The verb "increase" which appears in the definition must be taken in its dynamic sense. It is with respect to the growing series that a limit is properly limit, and not with respect to some result of the growth. It is only with respect to the series as moving toward the limit that the limit is properly limit. No variable has a limit because of some value which is very close or closest to a limit, but because of the possibility of an ever closer value. "Ever closer" has a dynamic meaning.

Now, when we speak of the "possibility of an ever closer value", it should be understood that the very possibility in question is defined by the "ever closer", that is by a fluent act, and not by some fixed term of the movement. This "possibility" is not defined by that which is merely act, but rather by that which is the act of that which is in potency as such. It is the possibility of getting

ever closer to the term, and not of actually reaching it.

We must be careful, however, not to consider the act of the movement itself as the term of the movement. This would be contradictory. For the limit is the term of the movement, and not the movement itself. All movement is toward something other than itself, just as any relation is "esse ad". And just as some relations are by nature such that they cannot "be in" that "toward" which they are, so some movement cannot actually reach the term "toward" which it is moving. (We shall analyse this point in the following chapter, and abide here by the mere indication of the distinction.)

The act which the movement can reach is an act which is closer, than some other, to that toward which the movement is tending, but the limit is never the limit of any such act, but of the "getting closer". Thus the limit is the term of "the act which is getting closer", as such. But the act which is getting closer is the act of the movement itself. Therefore the limit is the term of the movement "qua getting closer", in such a manner that if the "getting closer" ceased, it would no longer be the term; whereas, the term of movement in the ordinary sense must be defined by the possibility of actually reaching it, whether it is reached or not.

The variable is ordered to a limit only in so far as it is actually manifesting less and less op-