

Article

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Some Aspects of the Infinite

"Hence many of the topics which used to be placed among the great mysteries — for example, the natures of infinity, of continuity . . . — are no longer in any degree open to doubt or discussion. Those who wish to know the nature of these things need only read the works of such men as Peano or Georg Cantor."

B. RUSSELL (*Mysticism and Logic*).

"The infinite, like no other problem, has always deeply moved the soul of men. The infinite, like no other idea, has had a stimulating and fertile influence upon the mind. But the infinite is also more than any other concept, in need of clarification."

D. HILBERT (*Über das Unendlich*).

The difficult and bewildering question of the infinite is present, one way or another, on every level and in every field of knowledge. But it is undoubtedly in mathematics that the infinite holds the largest sway. "If in summing up," writes Hermann Weyl,¹ "a brief phrase is called for that characterizes the life center of mathematics one might well say: mathematics is the science of the infinite." And if one must admit the unparalleled importance of the part played in mathematics by the infinite, one must also admit the paradoxical character of this part. Indeed, to banish the infinite from mathematics would be equivalent to reducing this science to a very primitive and dwarfish condition; the presence and the use of the infinite is responsible for the astonishing growth and the wonderful fruitfulness of mathematics. Let us recall only the infinite processes which, once mastered, have made it possible for mathematics to become the unrivaled instrument for the investigation of the world of quantity. However it is no less certain that the infinite has been, for the mathematician, an ever present trouble-maker and has given him the worst nightmares; in fact, it alone is responsible for the most severe crises ever experienced in mathematics, crises shaking the very foundations of this science.

1. *Philosophy of Mathematics and Natural Science*, Princeton, Princeton University Press, 1949, p.66. A few years earlier, Weyl had also written: "Mathematics is the science of the infinite, its goal the symbolic comprehension of the infinite with human, that is finite means." (*The Open World*, New Haven, Yale University Press, 1932, p.7.)

The discovery of irrational quantities by the Pythagoreans provoked the first crisis ; the obligation to insert the infinite in the very definition of irrational numbers is the origin of the difficulty. The second crisis followed upon the invention of infinitesimal calculus and the use of infinite processes. The third came during the second half of the last century ; it was caused by Georg Cantor when he publicized his theory of infinite sets and his transfinite arithmetic. The first two crises are now over. The third is still in its heyday with strong supporters of the Cantorian doctrine such as professor Abraham Fränkel of the University of Jerusalem.¹

This last crisis is likely to be the most severe of them all for the simple reason that it strikes at a few fundamental and primitive notions not only in the field of mathematics,² but also in the whole sphere of knowledge itself. Thus the Cantorian crisis means as much to everyone interested in the value of knowledge as such, as it does to the mathematician.

The great importance of this question incites us to investigate it. But it is too broad to permit a thorough treatment in a few pages. Accordingly, our study, reduced to a few points chosen among the most fundamental, will include two parts. The first one will deal with the significance and the foundations of the theory of Cantor ; the second will propose a short but critical examination of two points of the theory : the first of these points being the actual infinite ; the other, the famous statement, "the whole is larger than its part".

THE FOUNDATIONS OF TRANSFINITE ARITHMETIC

Cantor's theory is very often, and rightly so, called transfinite arithmetic. It is arithmetic because it is a mathematical science performing operations with numbers. However, since it deals with transfinite numbers or powers, it is different from elementary arithmetic which deals with finite numbers. Thus transfinite arithmetic

1. A. A. FRÄNKEL and Y. BAR-HILLEL, *Foundations of Set Theory*, Amsterdam, North-Holland, 1958, p.15. The authors admit that the difficulties caused by the theory of infinite sets regarding primitive notions have led to the "third foundational crisis that mathematics is still undergoing."

2. The difficulties caused by the transfinite arithmetic, in particular the antinomies to which it leads, have deeply influenced some mathematicians. One of them, Hermann Weyl, made this confession : "We are less certain than ever about the ultimate foundation of (logic and) mathematics. Like everybody and everything in the world to-day, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life : it directed my interests to fields I considered relatively "safe", and has been a constant drain on the enthusiasm and determination with which I pursued my research work." Quoted by Fränkel and Bar-Hillel in *Foundations Set Theory*, pp.4-5.

appears as a continuation of elementary arithmetic beyond the limits of the finite. Such enlargement of the frame of usual arithmetic presupposes an extension of numbers themselves.¹ Our problem and the difficulties it generates are related to this enlargement.

The extension of numbers is based upon and starts from the unlimited sequence of natural numbers 1, 2, 3, . . . Because it takes two different directions, it is better to say there are two extensions. The first of these extensions becomes necessary if one wants to meet the needs of mensuration, and to generalize the inverse operations of arithmetic as well, these being subtraction, division and extraction of roots. Hence it successively introduces whole numbers, rational numbers, irrational numbers which, added to the rational, give real numbers and, finally, if we remain on the elementary level, complex numbers. The second extension, similarly, starts from natural numbers as measures of finite multitudes, but takes the direction towards transfinite numbers as measures of the plurality of infinite sets. These two extensions can be distinguished and characterized in the following way : the first goes from the discrete towards the continuous within the finite, the other goes from the finite towards the infinite within the discrete.

We will deal only with the second extension — that towards transfinite numbers. Two problems are connected with it ; the introduction and the existence of transfinite numbers constitute the first, the second is concerned with the classification of these numbers. This classification, in turn, depends on the evaluation of the sets, the plurality of which is signified by these numbers.

The admission and rightfulness of natural numbers as symbols and measures of the plurality of finite sets corresponding to them brings

1. The renowned essay by Cantor appeared first in installments in *Math. Annalen*. Cantor made a book out of these parts ; it was published in Leipzig in 1883 under the title *Grundlagen einer allgemeinen Mannichfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*. For this occasion, Cantor wrote a preface including the following lines :

"The previous exposition of my investigations in the theory of manifolds has arrived at a point where its continuation becomes dependent upon a generalization of the concept of the real integer beyond the usual limits ; a generalization taking a direction which, as far as I know, nobody has looked for hitherto.

"I depend to such an extent on that generalization of the concept of number that without it I should hardly be able to take freely even the smallest step forward in the theory of sets ; may this serve as a justification, or, if necessary, as an apology for my introducing apparently strange ideas into my considerations. As a matter of fact, the undertaking is the generalization or continuation of the series of real integers beyond the infinite. Daring as this might appear, I can express not only the hope but the firm conviction that this generalization will, in the course of time, have to be conceived as a quite simple, suitable and natural step. At the same time, I am well aware that, by taking such a step, I am setting myself in certain opposition to wide-spread views on the infinite in mathematics and to current opinions as to the nature of number."

forth no difficulty at all : the very evidence of the existence of such sets constitutes their full justification. It is not so simple with transfinite numbers. According to their very function, they must signify and measure the plurality of infinite sets. But do such sets really exist? And if they do exist, where are they? Such is our problem in the last analysis.

It would be a step in the right direction if we could offer a first transfinite number and discover an infinite set. A first suggestion, which appears both simple and promising, immediately comes to mind : is it not possible to reach a first transfinite number if one starts with finite numbers and finite sets? Both constitute ordered sequences, related to one another by a one-to-one correspondence. The sequence of ensembles can be indefinitely extended ; whatever the last set reached may be, another one can always be reached or, if necessary, constructed, whose plurality exceeds by one element that of the preceding ensemble. Correspondingly, the sequence of natural numbers can always be continued : one needs but add a new number and that number corresponds to the last set reached or constructed. Thus the indefinite extension of both sequences would seem to supply us with a first infinite set and a first transfinite number. Unfortunately this road leads to a dead-end.

It is true that each sequence is unlimited and can forever be continued, but each set, each natural number is finite. No matter how far one has gone in the sequence of ensembles, only finite sets will be found. Similarly, the sequence of natural numbers will display nothing but finite numbers. But one might perhaps be more fortunate by considering the sequences themselves. Instead of considering only one set of the sequence, if one took all sets already given or constructed, would not the desired result be obtained?

The result would unfortunately not be obtained because all the ensembles reached or constructed are, taken together, but a finite set. If these sets may be said to be infinite, it is because these sequences can be infinitely extended. We do, indeed, discover there a kind of infinite, but it is a potential infinite. However, transfinite numbers are not associated with only potentially infinite sets. Cantor's transfinite numbers cannot but correspond to an ensemble the plurality of which is already completely given.

It is absolutely impossible to reach transfinite numbers through a continuous process from the finite towards the infinite. The infinite can be reached only through a daring jump which places one instantly on the shores of the infinite. One must boldly posit ensembles infinite in act, posit an actually infinite multitude, posit an actual infinite. As a matter of fact, that is the way Cantor meant things. That is why he hesitated so long before making the required jump into the infinite. This jump needed a great deal of courage, for it implied a break with the traditional conception of the infinite which then

existed among mathematicians and other philosophers or scientists : all were convinced of the impossibility of an actual infinite.

Cantor performed his jump into the infinite by choosing as the initial infinite set that of natural numbers. Not however as an open sequence capable of an infinite continuation — that would still be a potential infinite —, but as a set of which every element would already be actually given and determined.¹ To represent and measure this first infinite plurality, Cantor invented a new symbol out of zero and the first letter of the Hebrew alphabet, namely “aleph-zero” : \aleph_0 .²

This symbol represents the first transfinite number.

Such is the procedure followed by Cantor. Later on, we will examine the value of the result ; for the time being, let us consider a different point.

This other point is concerned with the evaluation of infinite sets and, consequently, with the classification of transfinite numbers. We are now in possession of a first transfinite number, but do others exist? If so, there arises the question of their classification according to their different pluralities, necessarily presupposing the evaluation and the mensuration of infinite ensembles.

But how does one proceed to evaluate infinite ensembles? By counting, by enumerating their respective elements, one might be tempted to suggest. The process is indeed excellent, but its sole drawback is that it cannot be applied in the domain of the infinite, because, there, it could never come to a close. Thus it becomes necessary to look for another procedure. This procedure will be found through the analysis of the process of evaluation in use in the finite domain.

The process we follow when we evaluate finite sets is far from being a primitive one ; on the contrary, it is a very evolved and elaborate one which, under cover of a remarkable simplicity of use, hides a no less remarkable perfection of structure. In order to grasp that more easily, let us start from a very primitive way of evaluation applied to a concrete example. Let us consider a room full of persons and of chairs. If every chair is occupied by one person and every person is seated, I should know immediately, *without counting*, that there are as

1. Abraham A. FRÄNKEL, *Abstract Set Theory*, Amsterdam, North-Holland, 1953, p.10 ; “In sharp contrast to this use of the word *infinite*, the set of all natural numbers considered above (as well as its scheme of order) is a proper, definite *actual infinite* : the set contains infinitely many elements each of which is well determined. There appears to be nothing absurd or contradictory in such a concept, constructed by a simultaneous act of thinking. As a matter of fact, concepts of this kind have been explicitly or implicitly used as long as mathematics has existed as a deductive science.”

2. Georg CANTOR, *Contributions to the Founding of the Theory of Transfinite Numbers*, trad. Jourdain, New York, Dover, n.d., pp.103-104 ; “The first example of a transfinite aggregate is given by the totality of finite cardinal numbers ν ; we call its cardinal number “Aleph-zero” and denote it by \aleph_0 : thus we define $\aleph_0 = [\nu]$.”

many persons as there are chairs, and conversely. That is the most fundamental element in all counting processes : it amounts essentially to the formation of pairs in such a way that one element is taken from the first set, the second from the other. In other words, one associates with one and every element of the first set, one and only one element of the other, and conversely. In his technical jargon, the mathematician calls that type of association a one-to-one or bi-univocal correspondence. Of what use can the examination of such a process be to us? From it we learn that these two sets are equivalent ; that is, they possess the same plurality of members, they have the same cardinal number.

But this process gives us an incomplete knowledge, leaving us half-way from the king of knowledge we seek, and hence it is insufficient in itself ; another one is needed. Clearly this first process informs us that a set has the *same* plurality as another one ; but, as such, it does not tell what *that* plurality is. Complete information is obtained simultaneously only if the plurality of one of the sets is already known. In fact, it is precisely what happens when we enumerate : for, we already possess a model-sequence which functions as a basis and means for comparison. It is an abstract and graduated sequence because we know the exact plurality of each number : that is the infinite sequence of natural numbers in which every term or number differs from the immediately preceeding one by a unit and represents the plurality of a corresponding set. When we enumerate, we do nothing but establish a one-to-one correspondence between the units of the set to be evaluated and those composing a number of the abstract sequence.

As concerns the infinite, one proceeds in an analogous fashion. We transpose the process into the domain of the infinite after effecting the necessary transformations. However one must not try to construct pairs progressively ; contrarily to what was possible in the finite domain, it is here impossible to construct a one-to-one correspondence step by step. In the infinite, the enumeration leads nowhere. In order to construct the one-to-one correspondence absolutely required for the evaluation of an infinite set between this set and another one of known plurality, the mathematician will turn master-magician ; he will invent and use all sorts of clever devices, but they will have to include only a finite number of steps. The example supplied by natural numbers and only even numbers illustrates the situation perfectly :

$$\begin{array}{ccccccc}
 1, & 2, & 3, & 4, & \dots & n, & \dots & (A) \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\
 2, & 4, & 6, & 8, & \dots & 2n, & \dots & (B)
 \end{array}$$

The (A) sequence includes the natural numbers, the (B) sequence includes only even numbers. To each natural number we associate its double, therefore an even number ; conversely, to each even num-

ber we associate its half. We have thus a one-to-one correspondence which can be stated in a finite way in the following law :

$$b_i = 2a_i$$

where a_i is an element in (A) and b_i the corresponding element in (B).

TWO DEBATABLE POINTS

This second part will be devoted to the examination of two controversial points : the first is concerned with actual infinity, the second with the statement "the whole is larger than the part."

1. Actual infinity

Cantor posited the actual infinite. The needs of his theory forced him to do so.¹ And, in so doing, he was well aware of the strong resistance he would have to encounter both inside and outside the world of mathematicians. The generally accepted opinion had always been and still was against actual infinity ; there was no antagonism towards the potential infinite, but the actual infinite always appeared as impossible to the great majority. And this majority includes such giant mathematicians as Gauss and Poincaré.²

1. Cf. the text quoted above, p.11, n.1.

2. Karl-Friedrich Gauss (1777-1855) wrote : "I protest . . . against using infinite magnitude as something consummated ; such a use is never admissible in mathematics. The infinite is only a *façon de parler* : one has in mind limits which certain ratios approach as closely as is desirable, while other ratios may increase indefinitely." Quoted by A. A. FRÄNKEL in his introduction to *Abstract Set Theory*.

Henri Poincaré shares Gauss' lack of enthusiasm. He writes : "Depuis longtemps la notion d'infini avait été introduite en mathématiques ; mais cet infini était ce que les philosophes appellent un *devenir*. L'infini mathématique n'était qu'une quantité susceptible de croître au delà de toute limite ; c'était une quantité variable dont on ne pouvait pas dire qu'elle avait dépassé toutes les limites, mais seulement qu'elle les dépasserait."

"Cantor a entrepris d'introduire en mathématiques un *infini actuel*, c'est-à-dire une quantité qui n'est pas seulement susceptible de dépasser toutes les limites, mais qui est regardée comme les ayant déjà dépassées."

"De nombreux mathématiciens se sont lancés sur ses traces et se sont posé une série de questions du même genre. Ils se sont tellement familiarisés avec les nombres transfinis qu'ils en sont arrivés à faire dépendre la théorie des nombres finis de celle des nombres cardinaux de Cantor."

"Malheureusement, ils sont arrivés à des résultats contradictoires, c'est ce qu'on appelle les *antinomies cantorienne*s. Ces contradictions ne les ont pas découragés et ils se sont efforcés de modifier leurs règles de façon à faire disparaître celles qui s'étaient déjà manifestées, sans être assurés pour cela qu'il ne s'en manifesterait plus de nouvelles."

"Il est temps de faire justice de ces exagérations. Je n'espère pas les convaincre ; car ils ont trop longtemps vécu dans cette atmosphère. D'ailleurs, quand on a réfuté une de leurs démonstrations, on est sûr de la voir renaître avec des changements insignifiants, et quelques-unes d'entre elles sont déjà ressorties plusieurs fois de leurs cendres."

Time is a slow but effective healer. It has lessened the bitterness of the controversies caused by the introduction of the actual infinite. But as a consequence of this relative truce, has the fundamental difficulty disappeared? Has the very possibility of an actual infinite become evident? Some people have very quickly solve the question : they are satisfied with a fancy jump or with a condemnation without appeal. Like F. Hausdorff in *Mengenlehre*, some still see, in this time-honoured opposition to actual infinity, mere prejudice or a "decree dictated by philosophers."¹

Others have more discriminating opinions. Abraham Fränkel is one of them. His opinion deserves consideration. Let us read a few lines taken from his book *Abstract Set Theory* :²

The chief purpose of the present book is to prove that and to show how, in spite of the authorities of more than two thousand years who have rightly or wrongly been summoned as witnesses against the possibility of actual infinity, *it is possible to introduce into mathematics definite and distinct infinitely large numbers and to define meaningful operations between them.* In showing this, we shall make plain that possibility of free creation in mathematics which is not equalled in any other science.

.....
In order to reach infinite sets, we have to consider the *creations of our thinking.*
.....

In sharp contrast to this use of the word *infinite*, the set of all natural numbers . . . is a proper, definite *actual infinite* ; the set contains infinitely many elements each of which is well-determined. There appears to be nothing absurd or contradictory in such a concept, constructed by a simultaneous act of thinking.

Fränkel feels obliged to justify the actual infinite and he tries to do so. This justification cannot be supplied by the physical world which, according to Fränkel, does not contain, as far as we know, any example of an actually infinite set.³ He discovers it in this somewhat magical power with which our intelligence is endowed and which enables it to produce or create a set that is actually infinite. Let us not plague Fränkel with questions as to what might be "a simulta-

Telle autrefois l'hydre de Lerne avec ses fameuses têtes qui repoussaient toujours. Hercule s'en est tiré parce que son hydre n'avait que neuf têtes, à moins que ce ne soit onze ; mais ici il y en a trop, il y en a en Angleterre, en Allemagne, en Italie, en France, et il devrait renoncer à la partie. Je ne fais donc appel qu'aux hommes de bon sens sans parti pris." (*Science et méthode*, Paris, Flammarion, n. d., pp.153-155.)

1. F. HAUSDORFF, *Mengenlehre*, 3rd rev. ed., New York, Dover, p.11.

2. Cf. pp. 3, 9, 10.

3. *Abstract Set Theory*, p.9 : "It thus seems that the external world can afford us nothing but finite sets."

neous act." What he means is probably this : through and in a single act of thinking, we have the mysterious power of creating and bringing forth a set which is actually infinite, that is, of well-determined entities, each being given in its proper and autonomous singularity. I am very grateful to Fränkel for his disclosing this marvellous power to me. But I would also have been very pleased if he had taught me how to use such a power. For despite my sincere efforts, I have never succeeded in creating an actually infinite set of ideal entities. I can very well form the concept of natural number, I can name it with a single name. But, by so doing, in no wise do I produce an infinity of natural numbers, each of them given in its own and proper singularity. In a unique concept, in a unique name, I reach all the possible individual numbers indeed, but only in what they share in common ; and, in this common aspect, they are undistinguishable. I cannot reach them in what is proper and particular to each of them. To reach any of them in what is proper to it, one needs as many distinct acts as there are distinct numbers. To reach an infinity of individuals in a universal notion is perfectly conceivable, but it would be a dangerous mistake to see there the creation of an actually infinite set with all the peculiarities proper to each element.

Among the authorities who, for two thousand years, have opposed the actual infinite and to whom Fränkel refers, are usually included Aristotle and St. Thomas Aquinas. But one must be prudent. To classify those two scholars unconditionally as absolutely hostile to the actual infinite would indicate a rather extended ignorance of their teaching. Aristotle indeed has absolutely rejected the existence of a physical body with infinitely large dimensions ; he has also refused to admit the existence, in the physical universe, of an infinite multitude. However he believed in the eternity of the world, thereby admitting that time and duration were infinite. But Aristotle never raised, in an explicit way, the problem of the possibility of an infinite multitude in act on the level of pure quantity and that of the spiritual world. Aristotle does not admit the infinite in the physical and material world. But, from there, it does not necessarily follow that the actual infinite is impossible absolutely and on all levels, in particular on the mathematical level.¹

1. ARISTOTLE, *Phys.* III, c.5, 204 b : "This discussion, however, involves the more general question whether the infinite can be present in mathematical objects and things which are intelligible and do not have extension, as well as among sensible objects. Our inquiry (as physicists) is limited to its special subject-matter, the objects of sense, and we have to ask whether there is or is not among *them* a body which is infinite in the direction of increase." On that subject, St. Thomas Aquinas has the following remark : "... ista questio quae est : an infinitum sit in mathematicis quantitativis et in rebus intelligibilibus non habentibus magnitudinem, est magis universalis quam sit praesens consideratio." (*In III Phys.*, lect. 7.) One must also read the following passages from St. Thomas : *Cont. Gent.*, II, c.81 and 92.

It might perhaps prove even more imprudent to count the Angelic Doctor among the absolute opponents of the actual infinite. What was his exact position? It is not easy to say. First of all, it is undeniable that, most of the time, he rejected the existence of the infinite in act, except for God. But it is no less certain that he never completely rejected its very possibility. Here are some proofs or, rather, signs. For the Arab philosophers, the number of human souls raised a difficult problem. They thought, first, that the universe and the generation of men never had any beginning; they admitted moreover that the human soul is immortal. Joined together, those two beliefs naturally lead to the question of whether or not the number of human souls is actually infinite or not. Avicenna and Algezelis believed that the number of human souls could be actually infinite; Averroës on the contrary thought it was impossible. Then, commenting the opinion of Averroës, St. Thomas writes: "Et hoc verius esse videtur."¹ Now, to consider an opinion as more true than its opposite, is not equivalent to consider as impossible that one which appears as less true; it is rather suggesting and retaining the possibility of its being true.

Again, at the very end of his treatise on the eternity of the world, the Angelic Doctor rather surprisingly writes that he is not aware of anyone having yet demonstrated the impossibility of an actual infinite: "Et praeterea non est adhuc demonstratum quod Deus non possit facere ut sint infinita actu."²

2. *Whole and part*

Cantor's arithmetic came into conflict with an old statement, namely "the whole is larger than its part": *totum est majus sua parte*. Till then, that statement has been accepted by almost all without contestation. Aristotle who was probably the first to give it its formulation, and St. Thomas in particular, mentioned it very often: they recognized in it a principle the truth of which is evident for all. The Cantorians reproach their forerunners with accepting as everywhere true a statement which proves true only on the finite level; for, they consider that the part can be equal to the whole in the infinite. Let us examine how they were led to introduce this restriction. We will question its value later on.

From the existence of one-to-one correspondence between two infinite sets, Cantor's theory concludes to the equivalence of the sets. Equivalent sets are, by definition, sets having the same plurality of terms or elements; in other words, there are as many elements in

1. *Quodl.* IX, q.1, a.1.

2. *De aeternitate mundi contra murmurantes*, ed. by Perrier. It would also be profitable to read many passages in St. Thomas. Among others: *Ia*, q.7; *IIIa*, q.10, a.3; *De Ver.*, q.2, a.10; *Quodl.* III, q.2, a.2; *In I Sent.*, d.43, q.1, a.1.

one as in the other, they possess the same cardinal number. Here is, for instance, a sequence of mutually equivalent sets :

1,	2,	3,	4,	n,	(A)
2,	4,	6,	8,	2n,	(B)
1,	4,	9,	16,	n ²	
10,	20,	30,	40,	10n	
10,	10 ² ,	10 ³ ,	10 ⁴ ,	10 ⁿ ,	

From the second one on, all those sets are proper subsets of the first, the set of natural numbers. Because of the existence of a one-to-one correspondence, they are all equivalent to the first one and, as a consequence, equivalent to one another. For our purpose, it is sufficient to retain the first two sets only. The first of them is made up of natural numbers, that is, both even and odd numbers ; the second includes only even numbers. Since even numbers constitute a proper subset of natural numbers, they look like a part with respect to the natural numbers which, then, look like a whole. But the (A) set is equivalent to the (B) set. Therefore the (B) subset is equivalent to the (A) set ; the part is equivalent to the whole. In other words, the natural numbers on the one hand, and, the even numbers alone, on the other hand, have the same plurality of terms, they are in equal number. Hence, the part can be equal to the whole.¹

To say the least, that conclusion is paradoxical. Is it an effect of magic or the conclusion of a rigorous proof? The case needs examination. Many considerations could and should be made on the subject, but we must limit them to what seems most essential.

a) First of all, we should not be surprised to see that it is possible to establish a one-to-one correspondence between natural numbers and even numbers taken alone, although it is impossible to do the same between a finite set and one of its proper subsets. Such a possibility holds for the infinite sets under consideration because they are open sets, that is, sets to which one cannot assign a last term. However large a proposed natural number may be, we can always find its corresponding even number : one has only to go far enough in the sequence of even numbers and, somewhere along the line, he is bound to discover the desired number which is the double of the first.

But, if such is the case, are we still permitted to consider natural numbers as being an actual infinite, a set where the terms are already given, as requested by Cantor? Do they not appear rather like a potential infinite, that is, a sequence to which it is always possible to add a new number, no matter how far we have gone? Because of these conditions, it is a little embarrassing to speak of a "whole" à

1. St. Thomas holds a different view ; according to him, there are more natural numbers than even numbers. He makes his position clear twice : *IIIa*, q.10, a.3, ad 3 ; *Quodl.* IX, q.1, a.1, ad 1.

propos of the set of natural numbers. Totality and potential infinity exclude one another. One significant difference between the actual infinite and the potential infinite is that, in the first case, the terms are simultaneously given without exception ; in the second case, they become given only in an unending succession. The set of natural number is of that kind, and the set of even numbers as well.

b) But let us admit, for a while, that it is permitted to speak of natural numbers as if they were a whole. Are we, correspondingly, permitted to consider even numbers as a part? It depends. Let us recall a few points. We previously had the following sequences :

$$\begin{array}{ccccccc}
 1, & 2, & 3, & 4, & \dots & n, & \dots & (A) \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\
 2, & 4, & 6, & 8, & \dots & 2n, & \dots & (B)
 \end{array}$$

On this basis, the Cantorians have concluded that the part can be equal to the whole. That conclusion is hardly admissible. And here is why. It is absolutely correct that even numbers are only a part of the set of natural numbers. But these even numbers which function as integral and constitutive parts of the set of natural numbers are not those even numbers belonging to the (B) sequence, but those which belong to the (A) sequence itself. In other words, the one-to-one correspondence from which the debated conclusion is deduced is not a correspondence between a whole and its part, but between two independent and autonomous sets. I am afraid there is a serious confusion made between the number, say, *two* as the object signified and, on the other hand, the mark 2 as the symbol of two ; that symbol can be materially repeated, but the signification remains unique and the same. Is it not that confusion that makes it possible to deduce the conclusion mentioned above? As a matter of fact, if one tries to establish a one-to-one correspondence between natural numbers, even and odd, of the (A) sequence and the only even numbers of the same (A) sequence, he soon realizes the impossibility of such a performance.

c) I wish to add a brief remark concerning the statement *omn. totum majus est sua parte* as considered by Aristotle and St. Thomas. According to them, it implies some restrictive conditions which it is not necessary to mention if one makes use of the terms in their strict sense. Whole and part are two correlative terms which can be used in more than one way. But, in the present statement, they are to be understood only of the integral whole and the integral part which are found more evidently and strictly realized in the quantitative domain to which the qualitative *majus* properly belongs. An integral whole reaches its integrity, its perfection, its completeness, when and only when it is achieved, when it has received all the parts required by its very nature. For Aristotle and St. Thomas Aquinas, the potential infinite, because it always remains incomplete, cannot strictly be called

a whole. Hence to the potential infinite could not be applied the statement *omne totum majus est sua parte*.

We do not know if there exists an actually infinite multitude — those proposed by Cantor do not seem to be authentic ones —, but if there are any, it seems that we should consider them as true wholes. And if they are true wholes of integral character, the statement “the whole is larger than its part” should hold true of them in the traditional sense.

The preceding remarks do not claim to close a debate. Their sole ambition is to supply a modest contribution to the clarification of this bewildering problem of the infinite, which ranks among the most elusive. The study of the infinite has been going on for centuries, and the end of this study will not be reached in a predictable future. It will always be a source of uneasiness and dissatisfaction for our intelligence. We may claim we have succeeded to master and domesticate it to a certain extent ; we shall never completely dominate it, for there exists a fundamental disharmony between the infinite and human intelligence.

Louis-Émile BLANCHET.
