RATIONALIZATION

IN MATHEMATICS

A DISSERTATION

SUBMITTED TO THE FACULTY OF PHILOSOPHY

OF LAVAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

Roman A. Kocourek

PROPOSITIONES

1. Inductio reducitur ad formam syllogismi.

2. Casus et fortuna non definiri possunt sine causa formali.

3. Sensibile externum recte dividitur in commune, proprium et per accidens.

4. Triplex datur principatus oeconomicus: despoticus, politicus et regalis.

5. Mathematicus procedit disciplinabiliter.

TABLE OF CONTENTS

Pages

SECTION I

THE PROBLEM OF THE CONTINUOUS AND THE DISCRETE

Chapter I

Statement of the problem 1

Chapter II

The problem among the Greeks 7

Chapter III

Examination of the terms of the problem 27

Chapter IV

Modern conceptions of the problem 41

SECTION II

THE NATURE OF NUMBER

Chapter I

Explanation of certain notions 51

Chapter II

Heterogeneity and homogeneity in number

1. The problem of number 56

2. Number is an unum per se 62

3. The ultimate unit gives form and unity to number 68

pages

4. Objections showing that number is not an unum per se 71

5. Objections showing that the ultimate unit is not the form of number 74

Chapter III

Numbering number 82

Chapter IV

Number among the modern philosophers of mathematics 86

I

THE PROBLEM OF THE CONTINUOUS AND THE DISCRETE

Chapter I

The quest for "mathematical Rigor" leads us to what Mr. A. Fraenkel calls the "central problem of the foundation of mathematics". (1) He states the problem as follows:

> Comment peut-on passer du domaine discret des nombres au continu, et cela de la façon la plus constructive possible, bien que la représentation d'un domaine sur l'autre se soit avérée positivement impossible? (2)

In the paragraph following this he gives the point of view to be taken in the discussion;

> L'abîme entre le domaine des nombres et le continu n'est pas, il est vrai, le seul qui doive nous occuper; il est devancé par l'abîme entre le fini et l'infini dénombrable, et, d'autre part, il y a des abîmes analogues aussi au delà du continu. Et pourtant, notre problème est unique en son genre. Car le passage du fini à l'infini discret, et, avec cela, le problème du transfini en général, peut être plus fondamental du point de vue philosophique; pour le mathématicien, ce n'est pas un problème mais une conditio sine qua non de son activité. C'est avec juste raison qu'un chercheur aussi finitiste que Weyl l'a affirmé: les mathématiques sont la science de l'infini. D'un autre côté, les abîmes au delà du continu ne présentent qu'un faible intérêt, du moins pour le présent. Jeter un pont sur l'abîme entre l'infini dénombrable et le continu, voilà ce que l'on doit faire si l'on veut que les mathématiques ne se séparent pas en deux "mathématiques" presque independentes.

Whether the "infinite" found in numbers is "real", "ideal", "potential" or "actual" makes very little difference, apparently. (3) Having stated the opposing conceptions of some modern mathematicians (4) on the nature of mathematical entities, he says:

> ...pour tous ces points de vue un même abîme, sur lequel il semble impossible de jeter un pont, sépare le caractère discret, qualitatif, individuellement différent de l'arithmétique (monde des nombres), du domaine continu, quantitatif, homogène de la géométrie (monde de la mesure).

Why it is impossible to have two independent sciences of mathematics, or why it is necessary to bridge the abyss between the numerable infinite and the continuum, is nowhere clearly stated. Moreover, from the terms in which it is stated and from some of his remarks, it is difficult to see whether it is the mathematician, the philosopher or the logician who is faced with this problem. In the beginning of the article he says:

> ...la relation entre continu et discontinu en elle-même merite d'être traitée toujours de nouveau; elle peut, certes, être considérée comme le problème le plus profond et le plus difficile de la construction logique de mathématiques; problème qui a resisté à plus de deux mille ans d'efforts des philosophes et des mathématiciens, de Zénon d'Élée à Poincaré, Hilbert et Herbrand. (5)

Mr. E.T.Bell, in a recent work, refers to the question in terms which at first sight seem to make it clear that the solution must be the work of the mathematician. He says:

> Counting by the natural numbers 1,2,3,... introduced mathematicians to the concept of discreteness. The invention of irrational numbers, such as $\sqrt{2}$. $\sqrt{3}$, $\sqrt{6}$; attempts to compute plane areas bounded by curves or by incommensurable straight lines; the like for surfaces and volumes; also a long struggle to give a coherent account of motion, growth, and other sensually continuous change, forced mathematicians to invent the concept of continuity.

> The whole of mathematical history may be interpreted as a battle for supremacy between these two concepts. This conflict may be but an echo of the older strife so prominent in early Greek philosophy, the struggle of the One to subdue the Many. But the image of a battle is not wholly appropriate, in mathematics at least, as the continuous and the discrete have frequently helped one another to progress. (6)

But this again is ambiguous. Why should a mathematician qua mathematician be concerned with concepts? Also, the fact that the continuous and the discrete have an opposition which is in certain respects like that of the One and the Many seems to indicate that there is here a metaphysical problem.

The same confusion is apparent when Mr. C. Boyer discusses these notions. He points out first that the discovery of the calculus

> had its origin in the logical difficulties encountered by the ancient Greek mathematicians in their attempt to express their intuitive ideas on the ratios or proportionalities of lines, which they vaguely recognized as continuous, in terms of numbers, which they regarded as discrete. (7)

Later, when he discusses attempts to give a rigorous foundation to the notions of the calculus, he says that some mathematicians and philosophers raised objections and arguments which

> were in the last analysis equivalent to those which Zeno had raised well over two thousand years previously and were based on questions of infinity and continuity. (8)

He then goes on to say that because of these objections <u>mathematicians</u> were led to examine the foundations of mathematics, and the rest of the text shows some of the attempts to give a "rigorous formulation" of the concepts of the calculus.

This general failure to connect the problem to some one discipline is in a way the result of the nature of the ~~xxx~~ question, as we shall see later. For the present it is sufficient to note the confusion in the various statements of it. Also, it should be noted, mathematical "rigor" in some way involves "throwing a bridge over the abyss between the numerable infinite and the continuum. (9)

At first glance, it seems strange to see so many of the mathematicians working on a question which is admittedly philosophic in at least some of its aspects. (10) What is even more strange is the almost universal skepticism concerning any possible resolution of the problem. Fraenkel, in his article, hopes for at least

> la comprehension mutuelle entre les differentes conceptions. (11)

While Pierpont says:

> Personally we do not believe that absolute rigor will ever be attained and if a time arrives when this is thought to be the case, it will be a sign that the race of mathematicians has declined. (12)

This same attitude of skepticism runs like a theme through the whole of Mr. Bell's Development of Mathematics.

Certainly the confusion shown in the statements of the problem is to a large extent responsible for this paradoxical situation. The blame for the confusion can, we think, be laid at the door of modern epistemological theories which have so mistaken this problem as to subordinate all others to it. But we can go back still further. The limited character of modern philosophical writing can be attributed to the historicist chicanery practiced today, often by well-intentioned individuals who simply lack the perspective to appreciate anything more than a century or two before their own period.

This last remark may seem odd, especially in view of the fact that today there is a greater interest in history and historical research than at any other time. But it is precisely this great interest in historical fact to the exclusion of historical ideas which is to be condemned. The self-confindent attack on the thought of their predecessors by men armed only with the instruments of historical criticism, languages, etc., has isolated modern philosophical thinking from its sources. There is today a wide-spread failure to comprehend Greek thought, not only in such attempts as the recent work of Mr. Bertrand Russell, but even in the much more carefully considered writings of such men as Cajori, D.E.Smith, Heath and others.

It is not our task here to examine in detail these errors and their reasons. (13) It will be sufficient if we examine the Greek writers themselves with regard to the question of the continuous and the discrete to find their statement of the problem and the solutions offered. Following this, we shall try to show how the failure to consider these solutions leads directly to the modern confusion.

Chapter II

Mr. Fraenkel said in his article that the problem goes back to Zeno the Eleatic. Let us then consider, for a moment, what Zeno was talking about. (1) The most important evidence about the nature and purpose of Zeno's discussion is found in Plato's Parmenides, 127-8. Here Zeno himself is shown as stating that his arguments are for the purpose of protecting the arguments of Parmenides proving that only the One exists. Thus he is arguing to show that Parmenides is not wrong for the reasons given by the adversaries. In other words, he is showing that the arguments brought against the doctrine of Parmenides involve contradictions.

The general form of Zeno's argument can be stated as follows: You maintain that the One, which is continuous, is divisible; and more than that, that it is divided and thus reality is plural. Then, let us suppose that the division has taken place. (You must admit this since you hold that things are plural.) Now, one of two positions is possible. Either you will have certain ultimate magnitudes which are indivisible and infinite in number; or your division leaves nothing at all. But either of these alternatives is absurd. In the first, a finite and divisible thing will be composed of that which is infinite and indivisible; while in the second, your plurality will be compsed of nothing.

The arguments of Zeno were answered by Aristotle. But before going on to these, let us note carefully the dialectical nature of the above argument. Zeno is not concerned to prove that Parmenides is right. He only proves that the argument of the opponent is not a good argument. Thus, such dialectic establishes a position negatively. The defender of a position, in this case Zeno, siezes upon certain ambiguities in the objection and thus destroys the objection. With this in mind, a careful reader will be able to see that all the arguments of Zeno exploit the same ambiguities.

When we examine the position into which the opponent was led by Zeno, we see that there are three notions which could be the cause of the ambiguity; the first is that the One is continuous; the second, that the continuous is divisible; and the third, that it is actually divided. Let us now examine Aristotle's ideas on these notions, which some have criticized as "vague" and others have claimed were "invented". (2)

Aristotle's definitions and arguments are found in the *Physics*. We shall begin with the definitions found in chapter 3 of book V, where he says;

> Next let us define the terms "together" , "apart", "in contact", "between", "in succession", "contiguous", and "continuous".

Before we go on, it should be pointed out that there is a certain order in the terms to be defined. (3) Thus, he will first define "together" because it is used in the definition of "in contact"; and "apart" is defined prior to "in contact" because, although it does not enter into the definition of the latter, it is the opposite of "together" and thus gives us a better notion of this latter term. Another thing to be noted is that, although these definitions occur here in the Physics, it would be a mistake to conclude that they apply exclusively to the objects treated by the natural philosopher. The fact that he here uses examples from both natural philosophy and mathematics should warn the reader that he is treating notions common to both sciences. Those who would charge Aristotle with confusion should read his careful distinctions given in other places. (4)

> Those things are said to be together in place when they are in one place in the primary meaning of the term; but those which are in different places are said to be apart. Those whose extremities are together are said to be in contact.

An important point to note in the above definition of "together" is the term "place", and that taken in its primary meaning as explained in book IV. (5)

The next notion to be defined is that of "between" which is used in the definition of "in succession":

> That is said to be "between" which a thing being naturally changed continuously, is innately apt to reach first, before it reaches that into which it is changed last. Thus to be "between" requires at least

> three things: for "last" is the contrary of a change. Moreover, that is moved continuously which lacks nothing, or at least very lettle, on the part of the thing; but it cannot lack anything on the part of time. For nothing prohibits there being a deficiency on the part of the thing as long as there is no deficiency on the part of time, but only of the thing in which it is moved; as the highest note sounding immediately after the lowest. This is evident both in those things which are moved locally and in other motions.
>
> The contrary according to place is the most distant along a straight line. For the minimum distance is finite: and measure is finite.

Here he first gives the definition. Then he explains that "last" is one of the contraries of a change. Thus, in order to have a "between" we need the thing which is between and the beginning and the end of the change. Next, since he uses the notion of "continuous change" and has not as yet defined this, he proceeds to do so. (6) Then he adds an explanation of what he means by "contrary according to place"; and because he uses here the notion of "straight line", he gives the reason for this.

Notice that this notion of "xxxxxx "between" is defined differently than the others. It is the one notion here defined which does not apply directly to mathematical things as well as to physical things. The reason for this is that this idea requires the notion of transition, and thus, since motion is found properly only in physical things, "between" will apply only indirectly to things which are not physical. (7)

> That is "in succession" which, when it is alone after the beginning either by position or by species or by something else so determined, has nothing between of the same kind as itself and that to which it is in succession: for example, a line between lines according as they are lines; a unit between units according as they are units; or a house between houses. There is nothing to prevent there being something of another kind between. For that which is in succession is in succession to a particular thing, and is something posterior: for one is not in succession to two, nor is the first day of the month in succession to the second: in each the latter is in succession to the former. The "contiguous" is that which, when it is in succession, touches.
>
> Since every change implies opposites, and opposition can be either that of contraries or contradictories, and since contradiction has no middle, it is clear that there will be a between in contraries.
>
> Moreover, the "continuous" is a species of the contiguous. For I say things are continuous when one term of each of those things which touch becomes identical and as the name signifies, is contained: this cannot be the case when the extremes touching are two. Such being the case, it is evident that the continuum is found in those thing from which some one thing is apt to be brought about through contact. And in the manner in which at some time something becomes one continuum, so also will the whole be one; for example, either by nailing or by glue or by touch or by being born thus.

Then, having defined these terms, he further clarifies them by comparing one with the other. First he compares "in succession" and "in contact", and secondly, "in contact" and "continuous".

> Moreover, it is clear that the first of these is that which is "in succession". For that which is "in contact" is necessarily in succession, but not everything that is "in succession" is in contact. Whence succession is found in those things which are prior by definition, as in numbers, but in contact is not.

> And if there is a continuum, it is necessary that things be in contact: but if something is in contact, it does not necessarily make a continuum. For it is not necessary that their extremes be one, if they are together: but if they are one, it is necessary that they are together. Thus insertion is the last in the order of generation: for contact is necessary if the extremities are to be naturally joined; moreover things that are in contact are not all naturally joined. But in those things in which there is no contact, it is clear that there is no insertion either.

After this Aristotle draws a corollary from what he has said:

> Hence if there is unity and point existing separately, as some say, it is not possible for unity and point to be the same. For in those things having points we find contact; but in units there is only succession. And for those things having points there can be a between; for every line is between points; but this is not necessarily true for units; for there is nothing between duality and unity.

Later we shall see that this fact, namely that there is nothing between duality and unity, has the greatest importance in the solution of the problem of the continuous and the discrete. Failure to accept this simple fact causes most of the modern difficulties.

The next consideration which is necessary in order to understand Aristotle's refutation of Zeno's arguments is found in book VI, chapters 1&2 of the <u>Physics</u>. Here he proves that a) no continuum is composed of indivisibles; and b) no continuum is indivisible. In order that the parts of the arguments might be more evident we shall place the text in an outline form, inserting appropriate headings.

1. No continuum is composed of indivisibles.
   a. These first proofs are more concerned with magnitude.
      1) He restates certain definitions:

         Now if that which is continuous, in contact, and in succession are as they were defined above (things being continuous whose extremities are one, in contact if their extremities are together, and in succession if there is nothing of their own kind between), it is impossible that something that is continuous be composed of indivisibles, for example, that a line be composed of points; if a line is truly continuous, and a point indivisible.

      2) He proves the proposition.
         a) He gives two main reasons.
            1a) The first reason has two parts.
               1b) The continuum is not composed of indivisibles either by contact or by continuation. Two arguments:

                  1c)
                     For the extremities of two points can neither be one, since there can be no extremity of an indivisible as distinct from another part, nor together, since there is no extremity of an indivisible because the extremity is other than that ~~which~~ of which it is the extremity.

                  2c)
                     Further, if that which is continuous is composed of points, these points must be either continuous or in contact with one another: and it is the same in all indivisible things. Therefore, on account of the reason given above, they will not be continuous. Moreover, everything that is in contact is so only if the whole is in contact with the whole, or part with part, or whole with part. But since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact

with one another as whole with whole, they will not be continuous: for that which is continuous has one part different from another and is divided into such parts diverse and separated in place.

2b) The continuum is not composed of indivisibles in succession:

Nor again can a point be in succession to a point or an instant to an instant in such a way that length is composed of points or time of instants: for things are in succession if there is nothing between of the same kind: but there is always a line between points and time between instants.

2a) The second main reason:

Again, if length and time were thus composed of indivisibles, they could be divided into indivisibles, since each is divisible into the parts of which it is composed. But no continuous thing is divisible into things without parts.

b) He clears up two doubts in his arguments,
1a) That there is a line between points and time between instants:

Nor can there be anything of another kind between points or between instants. For if there were, it is clear that that will be either divisible or indivisible: and if it is divisible, it will be divisible either into indivisibles or into things always divisible. But the latter is what we mean by the continuum.

2a) No continuous thing is divisible into things without parts:

Moreover, it is clear that every thing continuous is divisible into divisibles that are infinitely divisible. For if the continuum were divided into indivisibles, there would be an indivisible in contact with an indivisible: for the extremities of continuous things are one and are in contact.

b. The following proofs show that what was true of magnitude is also true of motion and time.

1) He states his intention:

The same reasoning applies to magnitude, to time, and to motion: either all are composed of indivisibles and divided into indivisibles, or none.

2) He proves the proposition.

a) As to magnitude and motion.

1a) He states the propostion:

This is evident, moreover, from the following. For if magnitude is composed of indivisibles, the motion which takes place in that magnitude will be composed of correspondingly indivisible motions.

2a) He gives an example:

For example, if the magnitude ABC is composed of the indivisibles A, B, C, each correspondi-ing part of the motion DEZ of O in ABC is indivisible.

3a) He proves the proposition.

1b) He states certain things necessary for the proof:

1c)

If, therefore, wherever there is motion there must be something that is being-moved through some part and wherever there is something being-moved there is motion, then being-moved will be composed of indivisibles. Thus O traverses A with its motion D, B with its motion E, and C with its motion Z.

2c)

If, therefore, it is necessary that that which is moved from one place to another is not at the same time being-moved and at the completion of the motion by which it moves when it moves (for example, if someone is going to Thebes, it is impossible that at the same time he is goint to Thebes and has gone to Thebes):

2b) He proves the proposition:

Therefore, O traverses the partless section A in virtue of the presence of the motion D. Whereby if its arriving at is after its coming, the motion will be divisible. For when it was coming, it was neither at rest nor had it passed, but it was between. Moreover, if at the same time it has come to where it is coming to, and is coming to where it is coming to, when it is coming there it will have come; and the completion of the movement is at the same place as the being-moved. But if something is moved according to the whole ABC, and the motion by which it is moved is DEZ; moreover, if nothing is being-moved through the partless section A, but has moved; motion will not be composed of motions, but of moments.

3b) Then he shows by three arguments that it is impossible for motion to be composed of moments:

1c)

Also something not-moved is moved: for that not-passing through A has passed through A. Whereby it will follow that something which never passed through has passed through: for that which does not pass through A has passed through A.

2c)

If, therefore, it is necessary that everything be either at rest or in motion, and if (the mobile) is at rest through each of the parts A B C; then there is something continually at rest at the same time that it is being moved. For it was moved through the whole of ABC, and it was at rest at any part; and thus at rest through the whole.

3c)

And if the indivisibles DEZ are motions, the thing would not be moved by the motion present but would be at rest.

Moreover, if they are not motions, motion would not be composed of motions.

b) As to magnitude and time.
   1a) The division of time follows the division of magnitude and e converso.
      1b) He states the proposition:

Similarly to length and motion, it would be necessary that time be indivisible and composed of moments existing indivisibly.

      2b) He proves the proposition by three reasons:
         1c)

For if everything is divisible, and if a thing with a constant velocity will pass through a lesser magnitude in less time, the time also will be divisible. Conversely, if the time in which something passes through A is divisible, A itself will also be divisible.

         2c) The second reason used the "faster" and the "slower".
            1d) He states certain things necessary to clarify the proposition.
               1e) How the faster is related to the slower in a greater magnitude.

****

2c) -----
   1d) -----
      1e) -----
         1f) He states the proposition:

Moreover, since every magnitude is divisible into magnitudes (for it was shown that it is impossible that some continuum be composed of atoms: and every magnitude is continuous), it is necessary that the faster traverse a greater magnitude in an equal time and a greater magnitude in less time, as those who define the faster say.

         2f) He proves the proposition:
            1g) The faster traverses a greater magnitude in an equal time:

For suppose that we have A faster than B.

Therefore, since the faster is that which is changed first, in the time in which A is changed from C to D, namely in the time ZI, B will not yet have attained D but will be short of it. Thus in an equal time the faster will go further.

2g) The faster traverses a greater magnitude in a less time:

Moreover, it will traverse a greater magnitude in less time. For in the time in which A has arrived at D, B being slower has arrived, let us say, at E. Then, since A has occupied the whole time ZI in arriving at D, it will have arrived at T in less time than this, say ZK. Now the magnitude CT that A has passed is greater than the magnitude CE, and the time ZK is less than the whole time ZI: thus the faster will traverse a greater magnitude in less time.

2e) How the faster is related to the slower in an equal magnitude.

1f) He states the proposition:

It is clear from this that the faster will traverse an equal magnitude in less time.

2f) He proves this by two reasons:

1g)

For since the faster traverses a greater magnitude in less time than the slower, and the faster considered in itself traverses LM, the greater, in more time than LX, the lesser, the time PR in which it traverses LM will be more than the time PS in which it traverses LX: so that, the time PR being less than the time PH in which the slower traverses LX, the time PS will also be less than the time PH: for it is less than the time PR, and that which is less than a first thing which in turn is less than a second thing, is itself less than the second thing. Hence it follows that the faster will traverse an equal magnitude in less time than the slower.

2g)

Further, if it is necessary that everything moved be moved in an equal, lesser or greater time than another; and if that which is moved in a greater time is slower, and that in an equal time, equally fast; the faster is not equally fast nor slower: nor will the faster be moved in a longer or an equal time. It remains, therefore, that it is moved in a lesser time. Thus, it is necessary that the faster traverse an equal magnitude in less time.

2d) He proves the proposition. (Cf. 2c) directly above.)

1e) He sets forth certain notions necessary for the proof:

Moreover, since every motion is in time, and in every time it is possible that something be moved; and since that which is moved can be moved faster or slower; the possibility of being moved faster or slower will be found in every time.

2e) He draws his conclusion from what has been said:

Since these things are so, it is necessary also that time be continuous. Moreover, by continuous I mean that it is divisible into parts which are always divisible. For, when we suppose the continuum to be such, it is necessary that time be continuous.

3e) He explains the proposition:

For since it was shown that the faster will traverse an equal magnitude in less time, let us take two mobiles, A the faster and B the slower, and let the slower be moved through the magnitude which is CD in the time ZI. Then it is clear that the faster will be moved through the same magnitude in a less time than this; and let this time of A be ZT. Moreover, while A which is faster has traversed the whole of CD in the time ZT, the slower traverses in the same time a lesser magnitude: thus, let this latter magnitude be CK. Moreover, while B which is slower has passed the magnitude CK in the time ZT, the faster will traverse the latter magnitude in less time: thus the time ZT is again divided. Moreover, when this is divided, the magnitude CK will also be divided according to the same reason:

and if the magnitude is divided, so also the time. And this process will go on forever, taking the slower after the faster and the faster after the slower and using that which was demonstrated. For the faster divides the time, but the slower divides the length. Therefore, if it is true that we can always make this conversion, and if division is always produced by the conversion; it is clear that every time is continuous. It is likewise evident that every magnitude is continuous: for both time and magnitude are divided by the same and equal divisions.

****

3c) He gives the third reason to show that time and mgx magnitude are similarly divided, considering one and the same mobile:

Further, moreover, it is evident from the ordinary way of speaking that if time is continuous, so also is magnitude. Because a thing traverses half the magnitude in half the time; and universally, less magnitude in less time. For the divisions of time and magnitude are the same.

2a) Finite and infinite are found similarly in magnitude and time.

1b) He states the proposition:

If either time or magnitude is infinite, the other is also; and in the way in which one is infinite, the other is infinite: thus if time is infinite with respect to its extremities, so also length is infinite with respect to its extremities; but if time is infinite in division, so also length is infinite in division; moreover, if time is infinite in both ways, length is infinite in both ways.

2b) He solves ax a doubt of Zeno: (8)

Hence also the reason of Zeno supposes something false, namely that it is not possible to traverse the infinite, or be in contact with the infinite points singly in a finite time. For length and time are called infinite in two ways, and in general any continuum; either according to division or according to the extremities. Therefore, it is certainly not possible to be in contact point to point with things which are infinite in quantity in a finite time, but it is possible to be in contact with those things which are infinite in division in a finite time; for time itself is infinite in this way. Thus also, in an infinite time, but not in a finite time, the infinite can be traversed; and infinite points can be contacted point by point in an infinite time, but not in a finite time.

3b) He proves the proposition.
1c) He restates the proposition to be proved:

Neither is it possible to traverse the infinite in a finte time; nor the finte in an infinite time: but if the time is infinite, so also the magnitude will be infinite; and if the magnitude is infinite, so also the time.

****

2c) He proves the proposition and its converse.
1d)Time cannot be infinite if the magnitude is finite, (two reasons):
1e)

Take a finite magnitude AB which is traversed in an infinite time G. Then take some finite part of the time, call it GD. In this time GD the mobile will traverse some part of the magnitude; let that part traversed by BE. Now this latter will either measure AB or fall short or excell it: it makes no difference which. For if the mobile always traverses a magnitude equal to BE in an equal time, and if this magnitude measures the whole; the whole time in which it will traverse the magnitude will be finite. For the time will be dividdd into equals as is the magnitude.

2e) This next reason is against those who would say that any part of an infinite time is infinite:

Further, if the mobile traverses only some magnitudes in an infinite time, but can traverse others in a finite time; take, for example the magnitude BE: this will measure the whole, and the mobile will traverse an equal magnitude in an equal time. Thus the time also will be finite.

1f) He shows the necessity for the reason in 2e):

It is evident that the mobile does not traverse BE in an infinite time, if we take some other mangitude which is traversed in a finite time. For if it traversed the part in less time, it is necessary that this part be finite, since AB is finite.

2d) He shows that the converse of 1d) is true:

Moreover, the demonstration is the same, if the length is infinite and the time finite.

****

2. No continuum is indivisible.
   a. He states the absurdity which would follow if this were not true:

   Therefore, it is clear from what was said that neither lines nor planes nor any other continua are atoms (i.e. indivisibles): not only for the reason here given; but if this were true the indivisible would be divided.

   b. Then he shows that this absurdity necessarily follows:

   For, since the faster and the slower are to be found in any time; and the faster traverses a greater magnitude in an equal time; moreover, it is possible for the faster to traverse a length which is double or one which is greater by one and a half (for this is the nature of the faster): therefore, let us take a velocity which is greater by one and a half than another in the same time, and let us take the magnitudes of the faster as divided into three indivisibles AB, BC, CD; and the magnitudes of the slower as divided into two indivisibles EZ, ZI. And thus also the time will be divided into three indivisibles: for the mobile will traverse an equal magnitude in an equal time. Therefore, let the time be divided into KL, to the question asked: for he asked if it is possible to traverse or number the infinite in a finite time: it does not suffice for the fact and for the truth. For if someone, leaving out the magnitude and asking about the same time itself, were to ask if it is possible to traverse the infinite in a finite time (for time has infinite divisions), this solution would not be sufficient.

But we will then have to set forth the truth of what we say by reasons. Anyone dividing the continuum into two halves does this by using one sign as two: for he makes it the beginning of the one and the end of the other. But it is thus that anyone numbering or dividing into halves proceeds. But, in using division of this kind, neither the line nor the motion will be continuous: for motion is continuous if it has some continuous aspect. Moreover, in the continuous there are infinitely many halves, not actually but potentially: but if these are made actual, the result is not a continuum but a stasis. This is very obvious because it occurs in numbering the mid-point: for it is necessary to number the one sign as two, (for it will be the end of one half, but the beginning of the other), if the one does not number the continuum, but rather the two halves. Thus we will say to the one asking if it is possible to traverse the infinite either in time or in length, that it is and it is not possible: for if they are actually infinite, it is not possible; but if they are potentially infinite, it is possible. For that which is moved continuously has traversed, in a certain sense, the infinite; but strictly speaking it has not: for it is possible to designate an infinite number of mid-points for a line: but the line itself in its substance and definition is something other than these infinite midpoints.

But has Aristotle, by these arguments, solved the problem? To answer this we must ask which problem is meant. He certainly does not seem to have solved the problem of Mr. Fraenkel and the moderns of "throwing a bridge over the abyss between the numerable infinite and the continuum". Rather, he has said that the line in its "substance and definition" cannot be composed of "these infinite mid-points". He has, however, shown that the problem of Zeno rests on false assumptions and contradictory definitions of the nature of the continuous, and a failure to perceive a necessary distinction between the divisibility of the continuum and the actual division of the continuum. Thus Zeno's problem, as stated, is shown to be a pseudo-problem which exploits an ambiguous definition of the terms involved. If it is this problem which "has resisted the efforts of philosophers and mathematicians for two thousand years, from Zeno of Elea to Poincare, Hilbert and Herbrand" then our work would be finished, --- Aristotle has given us the solution.

But this, we agree, would be an all too facile resolution of a difficulty which has had an enormous importance for modern mathematicians. And, although we shall find an ambiguity at the bottom of the modern difficulty, Mr. Strong is quite right when he says that the problem of the foundations of mathematics among the moderns takes a different point of view from that of the ancients. (10) However, the insistence by the modern thinkers themselves on the fact that in some say their problem goes back to Zeno leads us to consider the problem which occasioned the arguments of Zeno.

Zeno's "proofs" were directed at the Pythagoreans of his day (11) who held that all is number. It is supposed that for them, as well as for Euclid later, the definition of number involved the notion of the indivisible One or unit. (12) Thus if everything is composed of numbers, the continuum itself is composed of numbers, which in turn are composed of indivisibles. Here, then, we see the basis for Zeno's arguments. As we saw from Aristotle, to say that the continuum is composed of indivisibles is to open the door for endless contradictions.

But, if this is the problem the moderns have in mind, it is plain that Aristotle's solution is valid here also. The continuum is not composed of indivisibles and connot be. This led him, and Euclid afterwards, to distinguish two independent sciences of mathematics; arithmetic and geometry. Arithmetic treats of number which, in one respect, is a "multitude composed of units", and thus actually divisible into indivisible units; while geometry treats of the continuous which cannot be composed of units, but is potentially divisible ad infinitum. Therefore, it seems safe to conclude that it is not the abyss between this number and this continuum that the moderns wish to bridge.

Chapter III

One of the greatest difficulties is that the modern writers do not give us very clear notions of what they mean when using the terms, number, continuum, construction, etc. As one of them has said recently (1), "...mathematics is an exact science, but mathematicians use inexact terminology." When they speak of the continuum, they either assume that the "empirical" notion of continuity is so obvious that it is useless, if not impossible, to try to define it; or they maintain that the notion is so hopelessly confused and imaginary, so tied down to anthropomorphic representations, that it is something to be excised rather than probed. (2) The notion of number suffers from an equal lack of precise consideration. For most mathematicians, the natural numbers, so-called, are assumed as something known. (3)

Therefore, again, in order to find out just what is meant by the terms of the problem, we shall have to go back to the men who have worked at it and see just what they were trying to do. Also, because it seems that the significance of our problem was not fully realized until quite recently (4), we shall touch only briefly on the work of that large number of mathematicians who struggled with the problem before the nineteenth century. Moreover, this will mean again referring to the thought of the Greeks because of some of the serious misinterpretations existing today.

We are continually told that for mathematicians today, mathematics is not the science of quantity. Thus Cajori says:

> One of the phases of the quest for rigor has been the re-defining of mathematics. "Mathematics, the science of quantity" is an old idea which goes back to Aristotle. A modified form of this old definition is due to Auguste Comte (1798-1857), the French philosopher and mathematician, the founder of positivism. Since the most striking measurements are not direct, but are indirect, as the determination of distances and sizes of the planets, or of the atoms, he defined mathematics as "the science of indirect measurement." These definitions have been abandoned for the reason that several modern branches of mathematics, such as the theory of groups, analysis situs, projective geometry, theory of numbers and the algebra of logic, have no relation to quantity and measurement. "For one thing," says C.J.Keyser (5), "the notion of the continuum --- the 'Grand Continuum' as Sylvester called it --- that central supporting pillar of modern Analysis, has been constructed by K. Weierstrass, R. Dedekind, Georg Cantor and others, without any reference whatever to quantity, so that number and magnitude are not only independent, they are essentially disparate." Or, if we prefer to go back a few centuries and refer to a single theorem, we may quote G. Desargues as saying that if the vertices of two triangles lie in three lines meeting in a point, then their sides meet in three points lying on a line. This beautiful theorem has nothing to do with measurement. (6)

This universal rejection of the notion of quantity is paralleled by a similarly universal ambiguity as to just what this rejected "quantity" is. In this passage, for example, we have a confusion between the notion of quantity and one of its properties --- measure. Further, there is no explanation here or elsewhere in the book as to the nature of the "continuum" which has been "constructed" with no reference to quantity, nor to the nature of the "construction".

We refer again to Aristotle for an explanation of just what is meant by quantity. In book V of the Metaphysics, chapter 13, he defines a quantified thing:

> That is called a quantified thing which is divisible into constituent parts, each or every one of which is innately apt to be some one particular thing.

The last part of this definition, "...each or every one of which...", explains the kind of divisibility which is proper to quantity. (7) The notion of "constituent parts" can be understood in several ways. We have the "physical" parts of a thing, as matter and form, which Aristotle speaks of in the first book of the Physics; then there are potential parts, as the soul is divided into the intellective and sensitive parts; also we have the subjective parts, as the universal is divided into its inferiors. But by "constituent parts" here he means those integral and quantitative parts which are so composed that when they are divided each remains a complete unit in itself. An example of this might be the ordinary division of an amount of water into various receptacles, which takes place without any chemical change: each "part" of the water would be water, and a complete entity.

Following this, he gives the division of quantity into its species (8) and distinguishes its modes:

> Therefore, a multitude, if it is numerable, is a quantified thing; so also is a magnitude, if it is measurable. Moreover, that is called a multitude which is potentially divisible into non-continuous parts; and

> that a magnitude which is divisible into continuous parts. That which is continuous in one dimension is a measure in length; in two dimensions, width; in three, depth. Of these, moreover, finite plurality is called number; finite length, line; finite width, surface; finite depth, body.
>
> Furthermore, some things are said to be quantified in themselves; others, only with respect to something else: for example, a line is quantified in itself, but a musical thing is quantified with respect to something else.
>
> Of those things which are quantified in themselves, some are so by their substance, as a line is a quantified thing. For the quantified quiddity is found in the definition stating the "what is it". Others are properties and dispositions of such a substance; as many and few, short and long, wide and narrow, deep and shallow, heavy and light, and others like these. Moreover, great and small, and greater and smaller, used absolutely or with respect to each other, are in themselves properties of a quantified thing. But these latter terms are also applied to other things.
>
> Of things called quantified with respect to something else, some are so called in the way in which we said above that a musical thing or a white thing is quantified, i.e. because they are quantified by some quantified thing in which they are. But others are quantified with respect to something else as motion and time are. For these latter are said to be quantified and continuous because those things of which they are the properties are divisible. I refer here not to the subject of motion but to the magnitude in which motion takes place; for it is by reason of the quantitative nature of the latter that motion is quantified. Time, moreover, is quantified by motion.

However, it is not this physical, sensible, quantified thing which is the subject of mathematics, according to Aristotle. The mathematician studies quantity in abstraction from sensible things. This is explained in chapter 3 of book XIII of the Metaphysics. (9) And, lest anyone think that such an abstraction of a property from the thing itself involves error, he adds:

> Thus, if we consider certain properties in separation from other attributes, and consider their properties; when we so consider these objects, we shall not be in error by reason of the abstraction made, any more than he who draws a line and calls it a foot long when it is not; because the error is not included in the premisses. The best procedure in each inquiry is to consider separately that which does not exist in separation; which is what the arithmetician or the geometrician does. (10)

Therefore, when he says that mathematics treats of quantity, he is speaking of quantity considered in abstraction from sensible things. If we define this quantity as such we would say that quantity is that according to which a quantified thing is divisible into homogeneous parts. But then we see immediately that there are two kinds of quantity. There is one kind of quantity, number, which is divisible into homogeneous parts (units) which are not further divisible because they do not have the same nature (plurality) as the whole which they constitute. Then there is another kind of quantity, magnitude or continuous quantity, which is divisible into homogeneous parts (magnitudes) which are further divisible because they always have the same nature (magnitude) as the original whole which they constitute.

It was because of this evident distinction, found in the nature of quantity itself, that the Greeks were led to distinguish two sciences of mathematics: arithmetic and geometry. In the investigation of the properties of number and those of the contunuum we see a confirmation of this radical diversity.

The qualities which are applied to triangle, line, cube, etc., are very different than those of number, as odd, even, prime, etc. And, while it is true that we call certain numbers square, rectangular, etc., this is only in imitation of those qualities which are properly found in continuous quantity. (11) The same diversity is found when one quantity is compared to another in the same genus. The relation found between numbers will always be commensurate, while that between continua will be sometimes commensurate and sometimes incommensurate because of the indefinite divisibility of the continuum.

When we turn now to the Elements of Euclid, we see that this division of mathematics is rigorously adhered to. He carefully defines the kinds of quantity he treats in the parts of his treatise. Thus, for the first six books, we have definitions which use the terms "magnitude", "line", etc. Also the theory of proportion found in geometry is treated differently than that found in arithmetic.

In his commentary on the text of Euclid, Heath makes a criticism which indicates his misconception and confusion of the ideas clearly set forth by Aristotle and Euclid:

> It is a remarkable fact that the theory of proportions is twice treated in Euclid, in Book V. with reference to magnitudes in general, and in Book VII. with reference to the particular case of numbers. The latter exposition referring only to commensurables may be taken to represent fairly the theory of proportions at the stage which it had reached before the great extension of it made by Eudoxus. The differences between

the definitions etc. in Books V. and VII. will appear as we go on: but the question naturally arises, why did Euclid not save himself so much repetition and treat numbers merely as ~~xx~~ a particular case of magnitude, referring back to the corresponding more general propositions of Book V. instead of proving the same propositions over again for numbers? It could not have escaped him that numbers fall under the conception of magnitude. Aristotle had plainly indicated that magnitudes may be numbers when he observed (Anal. post. 1. 7, 75 b 4) that you cannot adapt the arithmetical method of proof to the properties of magnitudes if the magnitudes are not numbers. Further Aristotle had remarked (Anal. post. 1. 5, 74 a 17) that the proposition that the terms of a proportion can be taken alternately was at one time proved separately for numbers, lines, solids and times, though it was possible to prove it for all by one demonstration; but, because there was no common name comprehending them all, namely numbers, lengths, times and solids, and their character was different, they were taken separately. Now, however, he adds, the proposition is proved generally. Yet Euclid says nothing to connect the two theories of proportion even when he comes in X. 5 to a proportion two terms of which are magnitudes and two are numbers ('Commensurable magnitudes have to one another the ratio which a number has to a number'). The probable explanation of the phenomenon is that Euclid simply followed tradition and gave the ~~theories~~ two theories as he found them. This would square with the remark of Pappus (VII. p. 678) as to Euclid's fairness to others and his readiness to give them credit for their work. (12)

When Aristotle "plainly indicated that magnitudes may be numbers" he also indicated that this consideration was according to common principles and not according to the principles proper to magnitudes and numbers, since he very specifically insists on this distinction in other places. Heath might have been warned by the fact that "there was no common name comprehending them all...and their character was different". That this community is not sufficient for demonstrative science is shown in a further passage in the same work (Anal. Post. 1. 11, 77 a 26), where Aristotle says:

> All sciences communicate (or participate) in the common principles. Moreover, I call those principles common which the demonstrating sciences use as something from which (the demonstrations proceed); but they do not demonstrate about these (as subjects), nor are they what (as conclusions) they demonstrate.
>
> Also dialectic is about all these (as subjects) as is any other science when it attempts to demonstrate common principles, such as that it is impossible to affirm and deny the same thing of the same subject, that equals subtracted from equals leave equals, and others that are like these. But dialectic does not have a certain group of things defined in this way, nor does it belong to any one genus of objects. Otherwise it would not proceed by interrogation, because it does not demonstrate that the same thing belongs to things opposed. But this was shown in what was said about the syllogism.

Therefore, while there is now a common demonstration for the alternation of proportion, this is dialectical insofar as it is common to both discrete and continuous quantity. To conclude, as Heath does, that magnitudes may be numbers is to reduce the sciences of mathematics to dialectics. Aristotle goes to some length to point out the distinction between science and dialectic in IV Metaphysics, chapter 2, 1004 b 17:

> Dialecticians and sophists present themselves under the same guise as the philosopher, for the sophist has the appearance of the philosopher, and the dialecticians argue about all things: being (the subject of the philosopher) is common to all things. And, moreover, they argue about these because these common things are proper to the philosopher; for sophistic, dialectic and philosophy treat the same genus of objects.
>
> But the philosopher differs from the dialectician in the efficacy of his reasoning, and from the sophist in his outlook on life. The dialectician arrives at probable

> conclusions concerning those things about which the philosopher demonstrates, and the sophist appears to be a philosopher but is not.

Therefore, it is plain that for Aristotle a demonstration which proceeds, not from the proper principles of the subject under consideration, but rather from certain principles which the subject has in common with other things, is not scientific but dialectical. Also, because such a procedure is dialectical, it can never attain the proper reasons of the subject but only the common or extrinsic reasons. Because of this, its proof is only probable. But Aristotle has shown in many places that for him the mathematical sciences are sciences in the strictest sense. More than that, they are the science most proportionate to our way of knowing. Only in this science does the mind find an object which is at the same time more knowable for us and more knowable in se.

Moreover, the radical diversity between discrete and continuous quantity demands that any treatment according to the proper principles of the subject recognize the fact that there are two mathematical sciences: arithmetic and geometry. This diversity is shown by the fact that the division of quantity into discrete and continuous is accepted at the outset of the treatment of that category in the Predicaments. Since these two kinds of quantity differ generically, the proofs of the mathematician, if they are to be scientific, must apply to one or the other. Any "general" mathematics which would apply equally to both could only be dialectical. It is for this reason that Euclid was very careful to treat proportion or commensurability twice. In doing so he proceeded mathematically and not just dialectically. He wanted to know what commensurability was in quantified things, from the point of view of mathematics taken in its strict sense of science. Heath, on the other hand, seems to be so struck by his own interpretation of the method of proof used in Euclid X. 1, which is used to determine incommensurability, that he would throw out mathematics proper and content himself with this. He calls this method "the method of exhaustion". Since most of the modern errors on infinity, limit, etc., refer to this notion, it will be well to see just what it means.

At one point Heath seems to say that Aristotle has given us the method of exhaustion, while in another place he seems to deny that such a method was valid. On the one hand, Heath says: (13)

> ...Aristotle already has the principle of the method of exhaustion used by Eudoxus: "if I continually add to a finite magnitude, I shall exceed every assigned magnitude, and similarly, if I subtract, I shall fall short (of any assigned magnitude)." (VIII Phys. c. 10, 266 b 2)

But again he says:

> Aristotle's denial of even the potential existence of a sum of magnitudes which shall exceed every definite magnitude was, as he himself implies, inconsistent with the lemma or assumption used by Eudoxus in his method of exhaustion. We can, therefore, well understand why, a century later, Archimedes felt it necessary to justify his own use of the lemma. (14)

In his commentary on Euclid, Heath repeats his criticism of Aristotle's failure to appreciate the method of exhaustion and adds Archimedes' statement of the lemma (15):

> 'The excess by which the greater of two unequal areas exceeds the less can, if it be continually added to itself, be made to exceed any assigned finite area.'

Heath does not seem to have appreciated the text of Aristotle in III Physics, c. 6, 206 b 5-12, since he does not refer to it in this connection. There Aristotle says:

> If we take a determinate part of a finite magnitude and add another part determined by the same ratio (not taking in the same amount of the original whole), and so on, we shall not traverse the given magnitude. But if we increase the ratio of the part, so as always to take in the same amount, we shall traverse the magnitude, for every finite magnitude is exhausted by means of any determinate quantity however small.

For this reason it is difficult to see just what Heath means by the "method of exhaustion". However, it seems probable from some of his statements about Antiphon and Bryson (16) that he is criticising Aristotle for denying that it is possible to attain the limit of a series; for example, that it is possible by the use of inscribed, or inscribed and circumscribed polygons to finally reach the circle. If this is true, then for Heath the method of exhaustion means that by taking a determinate part of a finite magnitude and adding another part always according to the same ratio, and carrying this on indefinitely, we shall finally traverse a given magnitude.

But then there is a double confusion: a) as to Aristotle's statements that "the sum of the parts taken (in infinite division) will not exceed every determinate magnitude..." (III *Phys.*, c. 6, 206 b 19) and that "in respect to addition there cannot be an infinite which even potentially exceeds every assignable magnitude,..." (Ibid. b 20); and b) as to the nature of the proof used by Euclid in X. 1 and XII. 2.

As to the first point, Heath seems to have failed to see the difference between an addition which proceeds according to a fixed ratio and that which proceeds according to a determinate magnitude. In the one case we have the variable tending to the limit which can never be reached, while in the latter case we have simple addition which will exceed all limits. He seems to visualize the concept of limit after an empirical model and thus he says that such a process must come to an end. But this also was acknowledged by Aristotle in his critique of Anaxagoras in I *Physics*, c. 4, 187 b 30, where he says that division in physical quantity will not go beyond a certain magnitude. However, in quantity as it is considered by the mathematician we shall not reach the limit. This point has caused much difficulty for many who have failed to appreciate Aristotle's remark on the infinite in mathematics:

> Our account (of the infinite) does not rob the mathematicians of their proof by thus denying the existence of an infinite that is actually untraversable by addition. For they neither need such an infinite, nor do they use it: they state only that any straight line may be extended as far as they wish. That it is possible to divide in the same proportion as the largest quantity, another magnitude whatsoever. Whence, for the purposes

> of demonstration, it will make no difference whether the greatest quantity be finite or infinite: but there will be a great difference whether there is an infinite or not in the magnitudes existing in things. (Ibid. 207 b 27)

Here we have merely the recognition of the fact that no mathematical proof depends upon the existence of the actual infinite, as is clear from Euclid.

This brings us to the second misconception of Heath. In his commentary on Euclid XII. 2, he says:

> The first essential in this proposition is to prove that we can <u>exhaust</u> a circle, in the sense of X. 1, by successively inscribing in it ~~/~~ regular polygons, each of which has twice as many sides as the preceding one.

But what Euclid actually says, after describing the process of bisecting the sides of the inscribed polygon, is:

> Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle EFGH exceeds the area S.
>
> For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

To say that this will "exhaust" the circle is to state exactly the opposite of what Euclid has proposed. For in the event that it is possible to exhaust the circle, then it will no longer be possible to have "left some magnitude which will be less than the lesser magnitude set out".

Since Heath has mistaken both the subject of mathematics and the nature of its proof, it is not surprising to see him attribute this "method of exhaustion" to Archimedes. We would hardly expect him to understand the nature of the scientia media with which which the latter was occupied. As a matter of fact, Heath finds no difficulty in classifying Archimedes among those who were instrumental in the discovery of the calculus (17) and he passes over Aristotle's careful distinctions on the nature of demonstration in the scientia media (Anal. post. 1. 13, 78b 32) with statements like the following:

> In applied mathematics Aristotle recognizes optics and mechanics in addition to astronomy and harmonics. He calls optics, harmonics, and astronomy the more physical (branches) of mathematics, and observes that these subjects and mechanics depend for the proofs of their propositions upon the pure mathematical subjects, optics on geometry, mechanics on geometry or stereometry, and harmonics on arithmetic; similarly, he says, Phaenomena (that is, observational astronomy) depend on (theoretical) astronomy. (18)

This failure to grasp the precision of Greek thought is evident also in many of the modern historians of mathematics. (19) As a consequence, the problem of the continuous and the discrete has today an entirely different meaning. The notions of the terms of the problem have changed. In an attempt to find out what is meant by these terms today we shall examine some of the modern writers on the subject.

Chapter IV

According to many writers (1) the modern problem arises from the attempt to give a firm foundation to the calculus. The work of Newton and Leibniz is said to have raised again the antinomies of Zeno and caused a reexamination of the foundations of mathematics. The century from Newton and Leibniz to Cauchy and Gauss has been termed a "period of indecision", in which certain steps were taken but no final results achieved. In this period we find the attack by Berkeley criticizing the logic of the calculus. Because the theory of the method as proposed seemed to rest on the existence of a limit "ratio" $\frac{0}{0}$, and on the attainment of this limit, he stated the following "lemma":

> If, with a view to demonstrate any proposition, a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards destroyed or rejected by a contrary supposition; in that case, all the other points attained thereby, and consequent thereupon, must also be destroyed and rejected, so as from thenceforward to be no more supposed or applied in the demonstration. (2)

Included in this work were a series of questions directed at the mathematicians of his day, among which were the following:

> Where there are no increments, whether there can be any ratio of increments? Whether nothings can be considered as proportional to real quantities? Or whether to talk of their proportions by not to talk nonsense? Also in what sense we are to understand the proportion of a surface to a line, of an area to an ordinate? And whether species or numbers, though properly expressing quantities which are not homogeneous, may yet be said to express their proportion to each other?

The readiness with which the mathematicians took up this challenge indicates that, for them, this theory was an essential part of the science of mathematics. This failure to see the calculus for what it was, namely as an adjunct to the science, led many mathematisians into a philosophical discussion of the nature of a variable and a limit. Moreover, few if any perceived its philosophical character. They tended rather to regard the question as one falling properly within the province of mathematics. When we add to this the fact that measurement in physical quantity is not to be distinguished from a mathematical consideration of quantity abstracted from sensible matter, we begin to see the possibility for much confusion in any discussion of limits, variables, etc.

One of the first who took up these questions after the criticism of Berkeley was Benjamin Robins. He said that "the ultimate ratio of vanishing quantities" does not refer to some final ratio which would necessarily be $\frac{0}{0}$, as Berkeley had claimed, but rather referred to a "fixed quantity which some varying quantity, by a continual augmentation or diminution shall perpetually approach". (3) By this explanation he gave an adequate definition of the limit process in the mathematical consideration of quantity. But he leaves unexplained the justification for considering the limit as given in the determination of velocities, areas, etc.

D'Alembert also considered the problem from the point of view of mathematical quantity. He defined the infinitesimals of the calculus much as did Robins. He said that the symbols were used to avoid the lengthy expression of the limit concept. Moreover, he denied that there was an actual infinite and stated that the mathematician does not need the actual infinite for a valid proof. (4) In this latter notion he saw clearly the point about the infinite in mathematics brought out by Aristotle and Euclid. However, his confusion of the symbols used in expressing the tending of a variable to a limit with abbreviations shows a serious misconception of the nature of a symbol as opposed to an abbreviation.

Lagrange, one of the best mathematicians of the period, was mainly preoccupied with tests for the convergence of a series. He felt that it was possible to found the calculus on the development of a powers series. With regard to this development, Pierpont says:

> Lagrange's method of development of the calculus free from the knotty questions regarding infinitesimals and limits was received with considerable favor. It suffers however a mortal defect. It rests upon the assumption that a given function can be developed in a power series, and there is no known method of deciding this question independently of the thing he wishes to avoid, namely limits. (5)

The confusions growing up around the question led such men as Cauchy, Gauss and Bolzano to seek an "arithmetical" development of the notion of limit. Because they failed to see that a consideration of limit which bases itself on sensible quantity is quite different from that based on mathematical quantity, they were led to consider only number. It seemed to them as if any consideration of limit which based itself on continuous quantity would be subject to the difficulties arising from the question of the infinite. Actually, as we shall see later, it is only in sensible quantity that these difficulties arise and then they are not real difficulties because of the macroscopic nature of the point or limit in question.

Bolzano had shown that Lagrange's method was dependent on the idea of the limit of a series and he also showed that it was necessary, in infinite series, to determine whether these were convergent or divergent. In attempting to explain how a series could have a limit such that it could be applied to a continuous quantity he seems to have evolved the notion that the continuum is ultimately composed of points. (6) He disposed of the problem of an actual infinite by stating that a sign of an infinite aggregate is that a part of it can be put into a one-to-one correspondence with the whole. This use of a property as a definition, which is still accepted in some quarters today, leads to a confusion between an infinite series and an infinite aggregate.

Cauchy's definition of the limit concept is said to be founded solely on the notions of number, variable and function rather than on an appeal to geometric notions. (7) However, Jourdain maintains that we find a geometrical way of thinking in Cauchy's doctrine of limit and his notion of irrational number. (8) Both of these critics fail to say what is meant by "geometric" notions. Presumably they are notions involving continuous quantity either sensible or mathematical. Boyer, in the place cited, gives Cauchy's definition of a limit as follows:

> When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of the others.

How this involves or does not involve "geometric" notions is difficult to see.

The work of Gauss resulted in no clarification of the fundamental confusions. He is credited with being the "first to observe rigor in the treatment of infinite series" on the basis, apparently, for his work in the determination of criteria for convergent series. (9) However, any evaluation of the notion of limit solely in terms of convergence or divergence shows a lack of appreciation of the notion itself.

The many confusions existing in this attempt to "arithmetize" mathematics has been pointed out by Boyer:

> In spite of the care with which Cauchy worked, there were a number of phrases in his exposition which required further explanation. The expressions "approach indefinitely", "as little as one wishes," "last ratios of infinitely small increments," were to be understood in terms of the method of limits, but they suggested difficulties

which had been raised in the preceding century. The very idea of a variable approaching a limit called forth vague intuitions of motion and the generation of quantities. Furthermore, there were, in Cauchy's presentation, certain subtle logical gaps. One of these was the failure to make clear the notion of an infinite aggregate, which is basic in his work in infinite sequences, upon which the derivative and the integral are built. Another lacuna is evident in his omission of a clear definition of that most fundamental of all notions ---number--- which is absolutely essential to the definition of limits, and therefore to that of the concepts of the calculus. The first of these points had been touched upon by Bolzano, but the theory was not further developed until much later, largely through the efforts of Georg Cantor. In the second matter the difficulty is essentially that of a vicious circle in the definition of irrational numbers, and this Weierstrass sought to solve. (10)

To avoid a vicious circle in the definition of/numbers, irrational

Weierstrass gives a notion of number which supposedly does not employ the notion of limit. The concept of number itself

> is to be defined as the representation (Vorstellung) of the multiplicity (Vielheit oder Menge) of homogeneous parts (or elements). When we denote each of the homogeneous elements by the expression "one", the counting of the elements or units of the aggregate consists in ~~ka~~ the fixing of "one and one", "---and one", and so on, by new expressions: "two", "three", and so on. The number is the representation of the groups of elements denoted by these expressions. (11)

The numerical quantity is identified with the aggregate of its elements. When the sums of numbers taken a finite number of times are determined, there is little difficulty; but when the sum to be determined is that of an infinite number of numerical quantities, as a, b, c,..., a new definition must be found. The sum S of this latter is defined to be aggregate whole elements occur in one (at least) of a, b, c,...; and each of these elements e are taken a number of times n equal to the number of times that it occurs in a increased by the number of times it occurs in b and the number of times it occurs in c, and so on. In order that S be finite and determined, it is necessary that each of the elements which occurs in it occurs a finite number of times, and it is necessary that a number N can be assigned such that the sum of any finite number of the quantities a, b, c, ... is less than N. (12)

Although Weierstrass was very careful to restrict his "sum" to a finite quantity, his introduction of the notion of limit is evident in his last criterion. Moreover, by his inclusion of the notion of a "sum" of an infinite number of terms he made it easier for the modern theorists of number to confuse symbols imposed to represent number as a class of all classes that are similar to a given class and symbols imposed to represent the tending of a variable to a limit. We shall discuss this more in detail later.

It was the popularity of this theory of Weierstrass that led Poincare in 1900 to assert that mathematical rigor had been attained. Other thinkers, and Poincare himself later, were not so certain. The development of the theories of Dedekind and Cantor raised questions which are still being discussed.

Dedekind again insisted that arithmetic must be developed out of itself alone without reference to "geometric" intuition.

He held that it is possible to construct the irrational numbers in much the same fashion as negative and fractional numbers. (13) He finds the "essence of continuity" in the principle:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (14)

By the assumption of this axiom we attribute to a line its continuity, for, as he says immediately:

> If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, form filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle.

With these notions, Dedekind was convinced that there would be no difficulties about the "existence" of irrational numbers. However, since his "straight line" is composed of points, he should at least show us how he would answer some of the objections to this theory made by Aristotle. The fact that he speaks of the "continuity or discontinuity of space" in the same terms as the continuity or discontinuity of a line is also rather confusing. His notion, too, that if "space" were "discontinuous" it would suffice to fill the gaps "in thought" makes us wonder whether he is speaking mathematically or logcally.

In 1883 Cantor, while stating that his system was probably essentially the same as that of Weierstrass and Dedekind (15), undertook to set up a theory for the development of mathematics from number. At about this time the theory of Kronecker had become popular in certain circles. This latter theory stated that only the finite integers exist and that the rest of the structure is a system of relations. (16) Cantor refers to this idea by rejects it as erroneous and unproductive for science and mathematics. He passes over, also, Aristotle's notion of the infinite by saying that the arguments against the real existence of the infinite "refer back to an assumption which involves a petitio principii, namely that there are only finite numbers". (17) How thoroughly he misunderstood the Aristotelean notions is shown by the following passage:

> Mathematics is entirely free in its development and is only bound by the self-evident restriction that its concepts must be consistent with each other and stand in determinate (through definitions), orderly relationships to those concepts which have preceded, these being already present and established. It (mathematics) is obligated when new numbers are introduced to give definitions of them by which such a determinacy and, under conditions such a relationship to the older number is granted them, that they can in any given case be definitely distinguished from each other. As soon as a number xx satisfies all these conditions it must be regarded as mathematically existent and real. (18)

He then proceeds with his well-known development of the transfinite numbers according to his three principles of generation.

Cantor seems to have seen, to some extent, the implications in the earlier confusions. His insistence on the logical character of mathematics is a tacit renunciation of the science of mathematics and an acceptance of a pure dialectic to replace it. In regard to the development of number he spoke in an earlier work (19) of "the dialectic generation of concepts, which always leads further and yet remains free from all arbitrariness, necessary, and consequent" which he said he had arrived at ten years before. However, it is difficult to see how such a number can number the quantitative multiplicity in reality. If number is a pure ens rationis the search for a "rigorous" demonstration will still involve difficulties, as we shall see immediately.

With the development of the theories of Weierstrass, Dedekind and Cantor we find a tendency more and more to consider mathematics as a "logic". "Number" is to be developed from a few "logical constants". Then, from this "number" the "continuum" will be "constructed". This development has been carried out by the school consisting of Frege, Peano and Mr. B. Russell. We pass over, for the moment, the theory of number as expounded by this school. It will be taken up in detail after the discussion on the nature of number in the second part. Only by finding out what number is will we be in a position to understand the modern problem of "bridging the abyss" or even to understand why such an attempt is made.