II

THE NATURE OF NUMBER

Chapter I

Earlier, when we discussed Aristotle's definition of quantity, we indicated briefly that there were two distinct species of quantity, the continuous and the discrete. In order to see more clearly the nature of discrete quantity or number, some further considerations are necessary.

When we consider the whole in relation to its parts we see that there are certain things which form a homogeneous whole which is such that any part has the same nature as the whole itself. Thus, any part of a continuum is continuous; for, as we saw above by the arguments of Aristotle, it is impossible that any continuum be composed of indivisibles. Such a whole is found, to a certain degree, in physical things. There is a sense in which it is true to say that any part of water is water. However, as Aristotle points out in his criticism of Anaxagoras (I *Physics*, ch. 4, 187 b 23), the physical quantity of any thing cannot be divided beyond a certain magnitude. The division of water into "parts" will finally arrive at some point where we have the smallest "part" which is still water. Beyond this, any further division will give us parts which are not homogeneous with the whole.

This last division will divide a heterogeneous whole which is such that no part has the same nature as the whole. Thus, no part of a house is a house, no part of a man is a man. However, it is not necessary that the parts of a heterogeneous whole be of different natures from each other as in the examples given. A group of men is a heterogeneous whole because no part has the form of the whole. On the other hand, each part has the same form (humanity) as any other part. Thus we can have a heterogeneous whole with heterogeneous parts, for example, a house; or we can have a heterogeneous whole with homogeneous parts, for example, a group of men. (1)

Therefore, when we speak of a multiplicity certain distinctions are necessary. The multiplicity of parts can constitute a homogeneous or a heterogeneous whole depending upon whether they have, or do not have the form of the whole. If we call the latter sort of multiplicity heterogeneous, it will be necessary to distinguish between a heterogeneous multiplicity with heterogeneous parts and one with homogeneous parts. As we shall see, it is the latter sort of multiplicity which is found in number.

It is important to notice that the part, or unit, in the heterogeneous whole, or multitude, can be considered in two ways: either with respect to division or with respect to the multitude which results from the division. (2) When we consider the unit with respect to division, the unit implies privation: it is not the whole. If we consider the unit with respect to the multitude which it constitutes, it has the nature of a principle and has a relation to the multitude as to something principiated and derived from this unit: the multitude is made up of units. When the multitude is quantitative the parts are homogeneous. Therefore, in such a multitude the unit has a further relation to the multitude; namely that of measure to measured.

The reason for this is found in the homogeneity of the units. We have here not only plurality, but also "more and less". This "more and less" implies an order. When we prescind from order, multitude is confused. It is for this reason that there must be, in the quantitative multitude, the relation of measure to measured.

Measure certifies us about the determined quantity of some thing. (3) This is shown by Aristotle in X Metaphysics (ch. 1, 1052 b 19) where he says that "to be one"

> means especially "to be the first measure of each genus" and most properly of quantity; for it is from this that it has been extended to the others. For measure is that by which quantity is known primarily; quantity as quantity is known either by one or by number, and all number is known by one. Therefore all quantity as quantity is known by the one, and that by which quantities are primarily known is the one itself. Thus the one is the principle of number as number.

Therefore, measure is found primarily in quantity. Our notion of measure is gathered primarily from predicamental quantity and then extended to other things insofar as they are quantitative, either actually or virtually.

Among the species of predicamental quantity, measure is found primarily in discrete quantity. The reason is that it is only here that we have a "one" which is perfectly indivisible. In the others the "one" is not perfectly indivisible because it is not the "one itself" but something to which the one occurs. For example, the measure in length is one foot; in volume, one bushel; etc. Therefore, the most certain measure will be the "one" which is the principle of number, because it is absolutely indivisible. The measure in the other species of predicamental quantity will imitate this indivisibility insofar as it is possible. In scientific measurement, for example, we search always for ever smaller, and thus more precise, practical standards of measure. The dialectically perfect measure of length would have to be the point. It is for this reason that we say that there is no absolute measure in continuous quantity. The perfect measure found in discrete quantity will be the limit of the approaching indivisibility and certitude of the measure in the other quantities.

The measure in any genus will be that which is perfect in that genus; either absolutely or relatively. This is brought out by John of St Thomas when he speaks of eternity as a measure:

> Measure implies perfection, since we always take as measure that which is most perfect in each genus. Nor is it necessary that (measure) manifest the things measured according to the order found in imperfect know-

> ledge (namely our way of knowing); but only by way of something more simple and perfect by which the thing measured is reduced more to unity and uniformity. (4)

This perfection in the one, principle of number, is the reason for the attempt to reduce the continuum to number and thus to the one. It is of the nature of measure to draw all things to itself, insofar as it is possible. As John of St Thomas says:

> Insofar as a measure is more perfect it will be more perfectly united to that which it measures and it will draw the latter to itself insofar as it is possible. (5)

With these considerations in mind, we are now prepared to take up the question of the nature of number.

Chapter II

1. The problem of number.

In X Metaphysics (ch. 1, 1053 a 30) Aristotle says that number is "a plurality of units" and defines it (ibid. ch. 6, 1057 a 3), "a multitude measured by one". In terms of what was said above, number is a heterogeneous whole composed of homogeneous parts. But it will be necessary to consider some of the difficulties which were neglected in that earlier explanation. The first of these arises from the fact that number is a plurality.

There are different kinds of plurality making up different kinds of wholes. There is a whole which is accidentally one, such as a group of indifferently diverse things. Any plurality whatsoever will constitute at least a whole of this kind. On the other hand, there is a whole which is per se one. This last will be either a whole which is per se one substantially, or a whole composed of things which are distinct from one another but which are per se ordered. Again, this latter will be either a whole constituted by per se ordered formal differences, such as any group of numbers: the first ten odd numbers, the "perfect" numbers, the numbers with the same factors, etc.; or it will be a whole which is constituted by homogeneous terms. (1) Therefore, since number is a plurality, we must determine whether it is per se one or only accidentally one.

Hence, it will be necessary to go into this notion of per se unity more in detail. Whenever we speak of plurality, some division is implied. This division will be either formal or material. When, for example, the genus animal is divided into man and brute, we have a division by the form. Also the distinctions between numbers, as between 2 and 3, odd and even, prime and composite, are distinctions or divisions according to the form. On the other hand, when a continuous quantity is divided, we have only material division: all the parts have the same form.

From this distinction, it does not seem possible that the plurality of number is that of formal division. Formal division means a difference of form and a difference of form means heterogeneity of the constituent parts of the whole. But this is contrary to the definition of quantity. Hence, although different numbers are formally divided, the constituent parts of any single number must be homogeneous, i.e. the elements qua elements must have singly, and each distinctly, the same form. Yet, this similarity of form cannot be the formal principle of the unity of number. The inherent form of its elements is multiplied in the elements. Hence, this form is numerically many. But we are trying to determine whether or not any number is one per se or per accidens.

If we confined ourselves to the inherent form of the elements we would say that any number is but an unum per accidens. However, this would make number indistinguishable from a plurality whose elements are formally and indifferently diverse.

What, then, is the proper principle of the difference between the accidental whole of, say, point and horse, and the unity of two homogeneous elements of a whole? Surely the homogeneity is not indifferent to the type of whole it makes up, as is shown by the essential difference between the sum of formally different things, such as point and horse, and the sum of formally homogeneous things, such as horse and horse. If number is indifferent to the heterogeneity or homogeneity of its elements, it can never be but accidentally one. But this leaves its homogeneity unaccounted for. It does not explain how 1 horse + 1 horse = 2 horses, where "horse" is predicated univocally of each member of the whole, differs from the ineffable sum of 1 point + 1 horse. If we prescind from the formal difference and call the latter two, "terms", we would have either a purely logical univocity, leaving us with unity according to reason only, and, therefore, accidental; or we would have a really accidental whole. Hence, the whole remains simply accidental --- which gives us two different viewpoints manifesting the same thing, leaving the problem of the difference between our two sums.

The identity of the form of the elements is indifferent only when we prescind from the identity. Hence, accidental unity does not account for the peculiar whole constituted by homogeneous elements. It is not accidental that two horses are of the same species. The unity they thus constitute is formally different from an accidental whole. Now, since the accidental whole which we defined is characterized by its indifference to the form of its elements, and since this division between accidental and per se is adequate, number must be an unum per se. Yet, the problem remains: how can number be an unum per se?

For most of the modern philosophers of mathematics, since they either prescind from or deny the strictly quantitative nature of number or that it has to do with measure, these problems do not arise. They seem to be satisfied that number is only an accidental unity. Some of them specifically reject any attempt to determine such questions. Thus Mr. Russell says:

> But when we come to the actual definition of numbers we cannot avoid what must at first sight seem a paradox, though this impression will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive. (2)

Such a position makes it very difficult to argue the nature of number. With men of "prudence" of this sort, it will be necessary to show that a more careful examination of the nature of the "number" which they choose as fundamental for mathematics, does not even raise the problems they continue to confront. We will show this in a later part of the present work. The priority of the notions now under discussion will become evident when we show that the "number" of which the modern writers speak is dependent upon that which we are here trying to determine. Therefore, to bring out more clearly the nature of the unity of number, we we shall follw a discussion of the difficulties concerning this unity as found in John of St Thomas. (3)

During the period in which he lived --- the first part of the 17th Century --- attacks were already appearing on the nature of predicamental number as set forth by Aristotle; attacks which indicated a reversion to a Democritean idea already criticized by Aristotle. (4) Writers such as Fonseca, Suarez, Vasques, and the Conimbricenses taught that number is not an unum per se but only an unum per accidens. On the other hand, the disciples of St Thomas and Duns Scotus, while agreeing that number is an unum per se, disagreed in explaining this unity. John of St Thomas follows Aristotle and St Thomas. However, in his discussion of the problem he was still able to "suppose the common (vulgaris) distinction between numbering number and numbered number". (5) Numbering number is that in the intellect by which we number, while numbered number is the multitude itself which is numbered. Obviously, when he speaks of number, he is referring to numbered number which is found in things rather than to numbering number which is only in the mind. (6)

In the first part of his treatise he points out some of the possible difficulties to the unity of number. He syas that even among those who, distinguishing number from continuous quantity, hold that it is an unum per se in the genus of quantity, there is some difference of opinion as to just how it is one. To say, as St Thomas does, that number is one by a unity of order would seem to make number relative rather than quantitative. Order, considered in itself, pertains rather to the predicament of relation than to quantity. To add that it is the ultimate unit which gives the form and unity to number does not seem to solve the difficulty. For, although here the ultimate unit is considered in its order to the other units, it seems impossible to determine any unit in a number which would be more of the nature of ultimate than any other, since, ex suppositione all units are homogeneous.

Again, he says, since number is said to be discrete quantity, others (7) others attempt to find the unity of number in this very discreteness. Then the units which make up the multitude would be the subject and matter of this discreteness. But, if this is the case, it is difficult to see in what way discreteness is one form simply in the genus of quantity.

Also, it is difficult to see how the subject of this form could be so diverse and really separated as are the units in number. Moreover, discreteness seems to be nothing other than division; thus it would not be a form making a number one in species, but would only cause the multitude as such. But multitude as multitude cannot be a per se species of quantity.

Finally, he says, some (8) insisted that what is fundamental in the notion of number is the nature of measure. Thus insofar as number has this nature of measure it is one in the species of quantity. But this means that in reality number is not quantitative because the nature of measure is common to both corporeal and incorporeal multitudes. Those who would take this position will not be able to explain how number is a proper measure of quantity.

Having stated these preliminary difficulties, he proceeds to give his own ideas, basing himself on Aristotle and St Thomas.

2. Number is an unum per se.

He says that it is certainly the conclusion of Aristotle that number which is composed of quantitative units is a true and proper species of quantity. We saw this specifically in the passage quoted from V Metaphysics and the same doctrine is stated in the Categories. The fact that number is placed in a category indicates that Aristotle held that it has a proper per se unity. One of the requirements that anything be in a category is that it be an unum per se. Thus, if number is only an accidental unity it must be excluded from the category because in that case it has no per se definition; consequently, it does not have only one genus and one difference. This being so, it could not be placed in one category because a category is an ordination of genera and species by their differences. Moreover, Aristotle explicitly excludes the possibility of definitions for accidental beings in VII Metaphysics, ch. 4, 1030 b 4. (9)

He says that this same doctrine is affirmed by St Thomas in many places. (10) He adds that those who think that when St Thomas says that the ultimate unity constitutes the species of number the words are to be taken metaphorically, ignore the many places where he teaches absolutely that number is an unum per se and is placed in the predicament of quantity. He then cites the very explicit text from the commentary on VII Metaphysics:

> Duality is not two unities, but something composed of two units; otherwise number would not be truly and per se being, but per accidens. (11)

For the analytical reason of the per se unity of number, John of St Thomas bases himself on the doctrine of St Thomas. (12) Number implies true extension and proper measure arising from the division of the continuum, which the multitude alone, taken confusedly in the manner of a heap or accumulation, does not. This is because number, founded upon divided quantity, has a true and proper measure according to extension and multiplication by which it can exhaust and adequate the parts and thus the extension of the continuum. Moreover, accidental quantities like time and motion which are quantified by reason of the magnitude in which they are, do not have this proper measure and extension. Further, number does not have this proper discrete measure by reason of continuous quantity; on the contrary, it divides and dissolves continuity. That the measure of number is a measure in the genus of quantity is evident from the fact that number has the capacity to measure according to the properties of quantity; i.e. by reason of equal and unequal, odd and even, and according to accretion and augmentation. Such measure is quantitative. Here he quotes St Thomas as saying that

> number, simply speaking, is that whose numbered parts make an aggregation. (13)

This, John of St Thomas says, is clear from the fact that by numbering and adding units we cause aggregation and increment in an ordered manner, i.e. with a measure of prior and posterior. Therefore, we have extension, an aggregated multiplicity with an order of the parts, and the only difference between this and continuous quantity is that in the latter the parts are united, in number the parts are separate. And, since number is constituted by units, we have a special kind of measure and extension. Therefore, if number is quantitative, it must be per se one, as was shown when we discussed plurality.

It might seem that such properties are found also in any multitude or collection. If we take two classes, for example, one of which comprises a point, a horse, and an idea; and another which comprises the first three with the addition of, say, a house; we might say that the addition of the house makes the second collection more than the first, and this is aggregation. Also, we might say that the first group is odd while the second is even. However, it is important to notice here that we do not have extension and aggregation arising from the division of the continuum, as we do in the case of number. The multiplicity in the example is that found in formal division. There is no homogeneity among the parts as we have in the case of number. Proof of this lies in the fact that there is no unit which is the principle of the multitude given in the example. If we say that the collection is "three" or "four", we will be asked immediately: "Three or four what?" The only possible answer would be: "Three or four things." But the term "thing" is applied differently to each; it is not univocal, or if it is, its univocity is only logical, as was pointed out earlier. Therefore, it is not true to say that one of the "units" will measure the multitude, because every measure must be homogeneous with that which it measures. This fact is recognized when we warn students in arithmetic that it is impossible to add, subtract, multiply or divide if the units are different in kind.

In the passage cited above, Aristotle says that "quantity as quantity is known ~~t~~ either by one or by number", indicating that in the predicament of quantity measure proper is found primarily in discrete quantity, because we measure the continuum itself by a numberation of the parts. Thus we see that number which is in the genus of quantity adds, over and above the multitude commonly so-called, the capacity of exhausting the mensuration of continuous quantity and making an aggregation and addition of one to another. This is not true of all multitudes, corporeal or incorporeal, although we can number any multitude by using numbering number, sometimes called abstract number, as we shall see later.

Another reason for maintaining that number is an ens per se and thus an unum per se is the fact that it is the subject of arithmetic. As the proper object of a science, it has a definition. This second reason, which is from effect to cause, is secondary, and in view of the widespread tendency today to consider mathematics solely as a game, probably will not carry much conviction. But such an attitude is disasterous. Mathematics has always been recognized as the science most proportionate to our mode of knowing. It is thus a touchstone for all methodological difficulties. If this science is now to lose its status, what happens to our knowledge? To try, as some do, to force mathematics to usurp the domain of logic has led some to try to force the human mind to adopt as its norm the mode of cognition of separated intellects.

There were some who opposed this argument that number must be an unum per se because it was the subject of arithmetic by saying that number is formally one by a unity of the mind which apprehends number as if it were some one thing. This sort of unity is supposed to be sufficient for it to be an object of science.

But, either that unity found in the mind is formally and per se in number or only accidentally as all natures, considered in abstactions, have a unity from the mind insofar as they are universal. The second is not ad rem because just as universality is a condition of the knowable object and is not the nature itself which is the knowable object, so also that unity of the mind will be only a condition. If the first is maintained, then number is formally an ens rationis. With this notion of number it is not very difficult to identify mathematics and logic. But, by so doing, it will be impossible to distinguish between numbered number and numbering number, concrete number and abstract number. Because, if number has no other unity than that of the mind, there will be no other kind of unity than that of numbering number. This was pointed out when we discussed plurality.

3. The ultimate unit gives the form and unity to number.

John of St Thomas next takes up the problem of the form by which number has unity. He says that that by which number is an unum per se is not the multitude of units or their aggregation or discreteness. This whole is only material in number and is, from this viewpoint, indifferent to per se or per accidens. As we saw in the discussion of plurality, this aspect alone does not show whether the multitude is one per se or per accidens. The per se unity of number is derived from the ultimate unit insofar as the many units are ordered under that ultimate unit. It is for this reason that the form of discrete quantity does not inform all the parts in the same way as the form of continuous quantity. The form of number orders the discrete parts, and they cannot be extended otherwise than according to the ordination under the ultimate unit. The unity of number is like the unity of place or the unity of a river. For these remain the same even though the bodies in a place are changing, or the water in the river is constantly changing.

That such a unity is per se and not per accidens is shown by St Thomas in his commentary on VII Metaphysics:

> Sometimes a composition is produced from many things so that the whole composed of many is a certain one, as a house is composed from its parts, and a mixed body from the elements. But sometimes the composite is produced from the many in such a way that the whole

> composed is not one simply, but only secundum quid; as is plain in a heap or pile of stones when the parts are actual, since they are not continuous. Whence, such a whole is simply many and one only secundum quid insofar as those many are associated with each other in place.
>
> The reason for this diversity is that the composite sometimes takes its species from some one thing which is either the form, as in a mixed body; or the composition, as in a house; or the order as in syllables or in number. And then the whole composed must be one simply. But sometimes the composite takes its species from the very multitude of the parts collected, as in a heap or in the people forming a state, and others of this sort: and in such the whole composed is not one simply, but only secundum quid. (14)

Therefore, since the multitude and discreteness in any number is material, and thus accidental, the unity in number is taken from the ordering of all these units under the ultimate unit which terminates them. Such termination and ordination is only possible because of the homogeneity in the units. For this reason, too, the order is not merely relative, but quantitative.

But here we have the difficulty mentioned earlier, namely in what way is the ultimate unit certain and designated if in any number, say, four, any unit can be the ultimate; and how is it that it only depends upon the designation by the intellect numbering? The answer is that in the thing the ultimate unit is given in its formal determination and designation which is the effect of terminating and enclosing the units, but it is not given in its material aspect. Four does not go beyond four units, otherwise it would not be four, and this effect is given in the thing. But that this effect come determinately from this or that unit is only material or accidentally any of the units could exercise this office. Thus it is one thing to be the ultimate unit as to designation, and another to be the ultimate unit as to the effect of termination; the second is given in the thing, the first can depend upon the act of designation of the intellect. We see something similar in the case of the circle. Since it is a finite and determined quantity, its circumference will have a beginning, middle and end; but these are not determined in all respects by its very nature and can be variously designated according to the various points on the circumference. Likewise the point in the middle of a line can be variously the beginning of one part and the end of the other, depending on the designation made at any particular time.

The reason for this is that the form giving unity in these things is not absolutely designated, but only respectively. That this part is the beginning or end, last or first, is said respectively, although it is clear that in the thing itself there is a given last and first, end and beginning, by the very fact that the quantity is finite. Thus we say that these are variable materially, but formally they are fixed, because the formal reason of ultimate has the nature of form, not matter; the others which are not ultimate always have the nature of matter and the determinable. But that which is called the ultimate unit denominatively and materially is variable in any particular denomination. It is for this reason that the ultimate unit is a partial and extrinsic form with respect to the other units. For, although with the others it composes number, it does not do so as if number were an accident composed of matter and form, but rather as a line determines a surface and a surface determines a body while all pertain together to an integral quantity.

John of St Thomas takes up several objections both to the notion that number is an unum per se and to the notion that the ultimate unit determines the form of the number. By following these objections, with his answers, we will be able to clear up some of the possible difficulties that could be raised to our position.

4. Objections showing that number is not an unum per se.

The first objection states that number is an accidental unity because the quantitative multitude is constituted by the division of quantity alone, all other forms are removed; but multitude alone cannot make an unum per se.

However, number is not constituted formally by division, rather division by itself is material as dissolving the continuum. We have number only when this division proceeds determinately in such a way that the parts are ordered under some ultimate unit as a measure. Therefore, just as for the continuum it is not sufficient to have any union of whatever kind, but rather one which is ordinative and terminative of the parts; so also for discrete quantity, separation and division alone are not sufficient, but the parts must be ordered to one ultimate unit and thus determined and made finite. It is only then that we have a special quantitative measure. It is only then that the six ones make one six.

Secondly, it seems impossible to have an unum per se when the multitude is composed of many things in act. It is for this reason that a white thing is called an accidental being. It is composed of white and substance, both of which are actual; although they are much more united as subject and form than are the units in number. Therefore, because number is composed of units actually divided, it must be an accidental being.

However, here again we point out that while it is true that many things in act cannot be an unum per se by the unity of form, still it is possible to have an unum per se by a unity of order and measure; just as in an artifact we have a unity, as one house, from many things in act. For per se unity is attributed to any thing which has its unity from form, composition or order, and this is opposed to accidental unity which derives from the multitude as such. As regards the example of the white thing, the argumen t should be carried further. For white taken formally is an unum per se and is definable by a unity taken from the form, even though denominatively it refers to a subject in which the white exists. But if we do not consider it formally, but rather the third thing constituted from whiteness and the subject, then it is called an accidental being. In the same way number formally indicates the ultimate unit under one order, but if we take it for that which is aggregated from the units and the order, it is an accidental being.

Thirdly, unity is opposed to multitude and destroys it. Thus if number is an unum per se, it cannot be a multitude.

However, here we say that the unity derived from continuity or form is opposed to multitude and division, but not the unity of order and discreteness as it ~~xt~~ is found in number under one ultimate unit terminating the order. Hence, the unity of order, found in number, presupposes a multiplicity, and this of a certain kind, namely homogeneous.

Finally, it seems that number must be an unum per se either with respect to the multitude or with respect to the discreteness. The first is impossible because multitude as such does not pertain to the genus of quantity, nor is it called a unity. The second seems similarly impossible because as discrete, number consists in the privation of continuity and nexus of parts. It is for this reason that Aristotle says in III *Physics*, 207 b 7, that number is many units, and in VIII *Metaphysics*, 1044 a 4, that number is not one but is like an accumulation.

However, when we say that number is a per se unity as discrete, we do not take discrete as privatively opposed to continuity, for thus it is nothing more than division. Rather, we take it ordinatively and formally insofar as it indicates discrete extension under one ultimate unit. And in III *Physics* Aristotle does not say that number is many units, but rather that it is a certain quantity and more than one, because any number is denominated by one. Thus in V *Metaphysics*, 1020 b 7, he says that the nature of each number is what it is once, as the nature of six is found in the fact that it is one six, and not in the fact that it is two threes. Also, in VIII *Metaphysics* he does not say that number is like an accumulation, but rather he says that if number is one, either it is an accumulation or we must explain in what way it is one. This he proceeds to do by pointing out that one part is as matter and the other, namely the ultimate unit, is as form.

5. Objections showing that the ultimate unit is not the form of number.

There are also difficulties as to how the ultimate unit specifies the number, and what the order of the units is. The first objection states that this order can only be in many subjects according to a relation by which they refer to each other and thus are ordered. Thus, this unity will be one of relation, not of quantity, and will pertain to the category of relation. To say that it is a transcendental relation rather than a prediecmental order, and thus can pertain to quantity, does not resolve the difficulty. As a transcendental relation its whole being would not consist in being related to some term but would have something else besides; as matter is related to form, but is something more than this relation. But if the order is found in such a relation, it will not give a quantitative unum per se; because either that quantitative order is the plurality itself of divided quantities or some diverse qua ntity supervening. In the first case, number will be only a multitude or an accumulation and not an unum per se. In the second case, it will be difficult to see in what way that quantity is in diverse subjects, or if it is in one, how it could affect all. Furthermore, the ultimate unit cannot be the form of number because either that ultimate unit specifies number mx because it is a unit, or because it is the ultimate. It cannot specify because it is a unit, because in that case even the first unit would specify the number; nor can it specify because it is the ultimate, because this designation is derived from the intellect. On the part of the thing no one unit is more ultimate than any other, and without a determined form there cannot be a determined species.

However, this argument overlooks the fact that the form of number which is the ultimate unit is not a form inhering in all the units; but still there is in the latter an order to the former. Nor is this only a relative unity of order; it is also quantitative. But it is not quantitative because of some quantity inhering in all the units, for thus it would be continuous quantity and number would have a unity of form and not of order. It is quantitative because it gives rise to a new kind of measure and of discrete extension; and this, which depends materially on continuous quantity as body depends on surface and on line, is called a quantitative order because formally it indicates a differnt measure of extension and aggregation. And therefore, number, insofar as it adds a special reason of measure which in a discrete manner exhausts continuous quantity, is found only in the genus of quantity. It is for this reason that the one which is convertible with being consists in the lack of division alone, while the nature of the one which is the principle of number consists in measure. For this reason the unity of order in number is distinguished from the unity of an army or of a city, which are accidental beings, by the fact that in the latter we find only the order of realtion, which is not sufficient for per se unity.

It is not sufficient because relation formally concerns many terms, and thus the unity of order is taken rather from plurality when it is only relative. But the unity of quantitative order, or of extension in number is not only a unity of relation but a distinction of units under one ultimate unit, under which the others are ordered and enclosed, so that they introduce a new way of measuring in extension itself and in discrete quantity. For the second part of the argument we say that that ultimate unit specifies number not only because it is a unit but because it is the ultimate unit. Moreover, the nature of ultimate in its material designation is derived from the intellect, and this does not specify number formally. But as to the effect and formality of ultimation, namely that all units are enclosed under a determined term, such that it does not remain an indeterminate and indecisive number, this is found in the thing. And when it is objected that without a determined form there is no determined species, the reply, from what was said above, is that without a determined form at least formally, there is no determined effect. But, if the formality is determined and the material designation is undetermined, we have simply a determined species in the thing; just as we have the unity of a river <u>in re</u>, although materially the water is not the same but now this, now that. (15) Still, because the river flows along the same course in the same succession with respect to the river-bed, it remains the same river. And it is the same in regard to the unity of place in succeeding diverse surfaces in the same substance. Thus the diverse units can succeed each other by designation of the intellect in the same reason of ultimate, and thus they have a unity in re formally, although not materially.

Secondly, it is not clear in what way the ultimate unit is united to the other units so as to make an unum per se with them. For it is not the intrinsic form, as was said, but is extrinsic to the other units. Moreover, how could a unit in Maine, for example, bex united per se with a unit in California so as to constitute one two?

However, this is not too difficult. Since any units in number are supposed as divided and separate, it will make no difference whether they are very far apart or whether they are near. They are not united by a union of continuation or copulation, but by a union of order and of discreteness, in which the one is related to the other by reason of a quantitative addition. Nor is it necessary here to look for another intrinsic form; for number does not have unity of form but rather the unity of quantitative order from discrete quantity, insofar as it introduces a new reason of measure formally in the ultimate unit and presuppositively in the others.

Thirdly, it seems that either the unit which becomes the ultimate is by itself the form of number or by something added. If by itself, then number adds nothing over and above unity, because any unit by itself suffices to constitute number, even though extrinsically other units are supposed. If by something added, this must be explained, i. e. whether it is a mode or a thing, whether it is found in all units as such or whether only in some one when it becomes ultimate. But even so, in the latter case, since it will have this designation from the intellect, we will not have something real resulting from this.

However, it must be noted that the unit as such is the inadequate and partial matter of number, but as ultimate it is the form. And further, the fact that it is ultimate as to the effect and formality of terminating and enclosing all the units under the term of a determined number is found in the thing, although the material designation is from the intellect. Also this termination or nature of ultimate is truly something in any determined number, because this office of enclosing and terminating the units under the nature of number, e.g., the nature of five or six, is truly found in the thing. Also this nature of term seems to be a mode by which the remaining units are terminated, so that they are coordinated and coextensive in the nature of discrete extension, which is a new termination. Nor can such a nature or mode be separated from the divided units by reason of the fact that they are determinately divided in act, because there is no determined division except by positing a term. And thus the ultimate unit under the formality of ultimate is not separable from the others with respect to which it is ultimate, unless we destroy the number and go from five, for example, to four.

Finally, it seems that if this ultimate unit be taken as removed from the other units, then either it would have that mode of number or that by which it is the form of number, or it would not have it. If it would have it, then the fact that it is ultimate and that it is the form of number comes to it from the intellect, and thus it is not something real.

However, in reply we say that the fact that the unit separated from the other units does not have the nature of ultimate arises not from a defect on the part of the unit so separated, but from a defect of the others which are necessarily presupposed and required in order that it be ultimate. And when it is inferred that to be ultimate is nothing other than a designation by the intellect, we refer to the distinction made previously: this is true as regards the designation of the material unit which is here and now taken as ultimate; but false as regards the effect and formality of terminating and enclosing the units under the nature of a certain measure and of discrete extension. And when it is said that we only require the positing of many units so as to have number, we say that materially or on the part of the matter we require only a plurality of units in number, but formally, namely so that number add a new reason of extension and discrete measure, something more is required; this is the order of units under one ultimate, terminating unit which is the quantitative order, as was explained.

Having thus gone into the nature of numbered number, the subject of arithmetic, we shall next consider how it is opposed to numbering number. This is necessary because of the fact that for the modern philosophers of mathematics only the latter is considered.

Chapter III

Earlier, when we pointed out that number is found only in the quantitative multitude, we said that the non-quantitative multitude could be numbered by numbering number, or abstract number. Thus the multitude constituted by point, horse, idea and house is said to be "four". From what has been said, it is clear that this four is not an unum per se as is four men, for example, because the ultimate unit in the first four cannot enclose and terminate the others as does the ultimate unit in four men. Still, it is plain that there is some kind of termination, because we call the multitude four, and not five or six. To find out just how we number such multitudes, we start from the statement of Aristotle in IV Physics, ch. 11, 219 b 5:

> Number is of two kinds: for we call that which is numbered and numerable, number; and also that by which we number.

Therefore, what is actually numbered or what is numerable is called numbered number, as ten men or ten horses, because the number is applied to numbered things. On the other hand, number taken absolutely, by which we number, as two, three, four, is called numbering number. (1) This latter is also called variously, absolute, simple, or abstract number, and sometimes the number of units. (2) It is that by means of which we count or number and is found only in the intellect; as opposed to numbered number which is in the things numbered. It is called absolute because it is number conceived outside of any subject, it abstracts from the possible subjects. It is simply the means by which we can attain predicamental number, and thus nothing other than the "reason of numbering in the intellect" (3). Therefore, it will be necessary to distinguish sharply between "two" as a "reason of numbering", and "two" which is predicamental number, i.e. a number applied to things, as "two men" or "two dogs".

We are now in a position to see in what way we "number" a non-quantitative or transcendental multitude. When we are faced with such a multitude we try to reduce it to some unit. Failing to find a true quantitative homogeneity in it, we look for some quantitative aspect. We saw above that it was of the nature of quantity to be divisible. Now, although this divisibility which is proper to quantity is found only in those things which are homogeneous in their parts, still we can consider other things as divisible and therefore in some way quantitative, which are only divisible or divided according to something extrinsic. Thus a virtue is said to be divisible and to have the nature of quantity by the nature and division of its acts and objects. Such quantity is called virtual. Furthermore, virtual quantity may be either discrete or continuous; for example, if we consider a virtue according to the number of the objects we have discrete virtual quantity, if we consider the virtue according to the intensity of the act with regard to the same object we have continuous virtual quantity. (4)

Now in the transcendental multitude we have divisibility, even actual division. However, we are able to consider the units in this multitude only with respect to the division, i.e. as undivided in themselves and divided from others. When we apply numbering number to this multitude we get a certain kind of measure. In absolute or numbering number the plurality has a kind of composition and aggregation which is less certain than the unit which is its principle; therefore, there is measurability of such a multitude not only as to the intellect but also as to the thing. (5) Such measurability, however, will not give us numbered number because we cannot attain a unit in the transcendental multitude which will formally enclose and terminate the other units. The units lack the quantitative order found in predicamental number. The point, horse, idea and house will be four ones only; while four men will be one four.

But here it might be objected that there can be no distinction between numbering number and numbered number, since the former, as well as the latter, is measured by one, which was the definition of number given above. Further, if it be said that numbering number is in the intellect while numbered number is in things; we point out that numbered number, as the subject of arithmetic, is also in the mind. As Aristotle said, mathematics treats of quantity in abstraction. Objects so considered cannot exist in reality, but only in the mind.

However, in reply we say that the measure of numbered number is according to quantitative order, while that in numbering number is not. In numbered number we have a multitude measured by one. The "one" here is the ultimate unit on the part of the number itself. Numbering number, on the other hand, does not measure the multitude constituting it, but another multitude, i.e. the numbering number is itself measure with respect to some measurable multitude and is therefore wholly extrinsic to the multitude numbered. It is not, as the ultimate unit of numbered number, intrinsic to the number in which it is the measure of the multitude. As to the objection that both numbering number and numbered number exist in the mind, we say that numbered number is in the intellect according to positive formal abstraction; whereas numbering number is from the mind as an object.

It has been the failure to consider this distinction between numbered number and numbering number which has led the moderns to their "logical" theories of mathematics. They treat, for the most part, with numbering number. By so doing, they find themselves faced with insoluble difficulties.

Chapter IV

In an article on Frege (1) H. R. Smart points out that for Frege numbers are neither physical entities nor do they refer to a psychical state or process. Rather,

> ...number is neither spatial and physical, nor subjective and mental, but non-sensible and objective, like the earth's axis or the center of the solar system (Grundlagen, sec. 26). Thinking, it seems plain to Frege, creates none of these 'objectives'; rather, they are eternally 'there', metaphorically speaking, to be thought about. Hence any attempt either to determine the nature of number genetically, or to trace the historical development of the number-concept, and in this wise to inquire into its possible derivation from more elementary but less precise ideas, Frege rules out <u>ab initio</u> as beside the mark. Like Bolzano's ideas-in-themselves, which Frege's 'objectives' quite closely resemble, mathematical entities are thus assigned to a special, sacrosanct realm of being, not subject to the vicissitudes of this earth, which the fortunate spectator may discover and contemplate, and, if constituted like Bertrand Russell's "free man", worship from afar. (2)

At first glance, it might seem that Frege was trying to describe the way in which a mathematical quiddity exists, and thus to be talking of numbered number. But in a later passage Mr. Smart shows that this is not true:

> Now it seems to be Frege's contention, though this is nowhere expressed with anything like the desirable clarity or convincing argumentation, that concepts, 'objectives', being neither mental nor physical, can therefore only be described as purely logical entities. It follows at once from such considerations, that number, as predicable of, or applicable to concepts, is also to be included within the general sphere of logic. Or, to put it in another way, since the subject matter of both logic and mathematics belongs to the same realm, they are to all intents and pruposes inseparable, if not identical sciences. (3)

Now, if number is a "pure logical entity" such that "the subject matter of both logic and mathematics belongs to the same realm", this cannot be what is described by Aristotle as numbered number. Such beings could never have a quantitative order such that by addition they would exhaust the quantity of any thing.

As a matter of fact, when Mr. Smart describes Frege's notion of number more in detail, we get a description, not of numbered number but of numbering number. In fact, it is so definite in this respect that it deserves quotation in spite of the length.

> ...in order to define numbers "we must clarify the meaning of the proposition, 'the number (Zahl) which applies to the concept F is the same as that which applies to the concept C'; i.e., we must reproduce the content of this proposition in another manner, without using the expression, 'the number (Anzahl) which applies to the concept F'" (Grundlagen, sec. 62). According to Frege it will be found that this clarification will further yield a general criterion for determining the equality of numbers, for grasping a determinate number as such, and for bestowing a proper name upon it.
>
> But to be able to assert the proposition just formulated is tantamount to being able to answer the question: when do the concepts (e.g., F and C above) applicable to two collections of objects (e.g., those 'falling under' concepts F and C) have the same number of terms --- or, as would ordinarily be said, the same extension? And the answer is: When there obtains a one-to-one relation between all the terms of the one collection and all the terms of the other, taken severally (such as that which holds, for example, of the collections 'husbands' and 'wives' in monogamous countries). Finally this "similarity" of two such collections --- to use Russellian language --- leads to the definition of the number of a given collection (extension of a given concept) as the class of all collections that are similar (stand in a one-to-one relation) to it; or, more pre-

cisely, 'the number of terms in a given class' is defined as the equivalent of 'the class of all classes that are similar to a given class'. Thus 'two' is 'the class of all couples'; 'three', 'the class of all triads', and so forth. Frege contends that this extensional definition follows from, and confirms, his view that numbers are to be predicated of concepts, that it yields the usual arithmetical properties of number, finite and infinite alike, and that it applies to '0' and '1', which are often treated as special cases, in the same manner as to all other numbers (cf. Grundlagen, sec. 62 ff.). (4)

Mr. Smart, in rejecting such a theory of number, does say:

> The leaves of a tree are possessed of various properties qua leaves, or intrinsically; but they are not numerable qua leaves, but rather only because and in so far as they are phenomenal objects qualitatively or intrinsically distinguishable from all other such objects, and at the same time only extrinsically or quantitatively distinguishable from, and related to, each other. (5)

But, while appreciating the necessity of a quantitative basis for number, he holds a notion of the science of mathematics which debars him from ever seeing the true nature of the mathematical consideration of quantity. For he says:

> From the point of view of mathematics as a science, and that means as a progressively developing body of knowledge, the very attempt to formulate a definitive definition of number is basically mistaken. No definition of any scientific concept can be more than provisional and temporary, and no scientific definition can have more than pragmatic sanction as a working instrumentality of the science at any given stage of its development. (6)

Here, besides appropriating the name "science" to that discipline which proceeds hypothetically, he confuses the nature of mathematical entities and natural beings which are quantified accidentally, as time and place. (7)

Since Bertrand Russell professes to follow Frege's def inition of number, it seems that he also refers to numbering number rather than to numbered number. This is further confirmed by some of his statements about number. (8) He says, for example, "A plurality is not an instance of number, but of some particular number." Again, he gives an operational definition of number which indicates that he means that by which we number rather than numbered number: "number is a way of bringing together certain collections, namely, those that have ~~the~~ a given number of terms." And, although he says that "it follows that the last number used in counting a collection is the number of terms in the collection, provided the collection is finite", he obviously is referring to the quasi-measure which is found in the unit measuring numbering number. For he adds immediately that "order is not of the essence of number: it is an irrelevant addition, an unnecessary complication from the logical point of view." Therefore, if "order is not of the essence of number", the "last number used in counting" will not order the other units under itself, making an unum per se. But this is precisely the case in numbering number, as was pointed out. Furthermore, he says:

> We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down.

Therefore, when he says that "the number of a class is the class of all those classes that are similar to it" or that "a number is anything which is the number of some class" he is referring to numbering number, if he is to make any sense at all.

Furthermore, the result of the modern "generalization" of number can only be numbering number. A sign of this is the fact that the term of the consideration in such a generalization is not number, but the operations performed with numbers. This has been noted by some of the modern writers:

> Negative numbers, imaginary numbers, quaternions, transcendental numbers, matrices, have been introduced into the domain of number because continuity and universality of treatment demanded them. But they have been designated as "numbers" because they share certain abstract properties with the more familiar instances of mathematical entities.
>
> Generality of treatment is thus an obvious goal of mathematics. But it is clearly a mistaken idea to suppose that the definition of "number" as applicable specifically to the cardinals 1, 2, 3, and so on, has in some sense been "extended" or "generalized" to apply to fractions, irrationals, and the rest. There is no generic definition of "number" of which the cardinals, ordinals, fractions, and so on are special instances except in terms of the formal properties of certain "operations". It is in virtue of the permanence or invariance of these formal properties that these entities are all "numbers". (9)

Also Peacock brings out this same point in his Algebra:

> The numbers which are the objects of Aristhmetical Operations, are considered as abstract so long as no specific properties are assigned to the units of which they are composed and concrete under all other circumstances, The notation, however, which we adopt for the purpose of representing numbers, generally suppresses all consideration of the specific properties of their component units, and consequently indicates no distinc-

> tion in the operations to be performed upon them, whether they be abstract or concrete. (10)

Such a generalisation of number is only possible if number is defined as the class of all classes having the same number. But this, as was seen, is numbering number. Therefore, any system which begins its consideration of number with this generalisation cannot arrive at the number of arithmetic. It is for this reason that we say that the problems of the modern mathematicians are insoluble. They are insoluble in the terms in which they are proposed. More than that, such problems do not even arise in a consideration restricted to numbering number alone.

If we restrict our notion of number to the number of a class, the problem of "throwing a bridge over the abyss between the numerable infinite and the continuum" is not a problem. More than that, such a problem cannot even be posed.

As we saw in the beginning, this problem presupposes a heterogeneity in numbers such that they are specifically different, and thus discrete. It further supposes that the continuum is homogeneous and thus opposed to the heterogeneity of number. But in order that our numbers be heterogeneous, each must have a unity in itself such that it is specifically different from any other. If number is a class of all classes similar to a given class, Mr. Russell has rightly said that the specific difference of any number cannot come from its nature, but must be derived from its relations with the other numbers. (11) But he has failed to see that taking number in its aspect of relation makes it an accidental being. If the number three has the nature of "threeness" only because it is greater than two and less than ~~four~~ four, three is only accidentally one. It is specified by that which is extrinsic to its nature. Similarly, when Dedekind says:

> If in the consideration of a simply finite system N set in order by a transformation ø we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ø, then are these elements called natural numbers or ordinal numbers or simply numbers ... (12)

the order in number is something extrinsic. The "distinguishability" here is the multitude considered as a multitude, and, as we saw above, as such it cannot be the principle of a per se unity in number.

It is true that Mr. Russell, with characteristic confusion, fails to see this similarity between his own definition of number and that of Dedekind (13); but this is clearly pointed out by Cassirer. (14) The latter has a clear view of the logic of the position that only numbering numbers exist, and draws his conclusions accordingly. As he says:

> What is here expressed is just this: that there is a system of ideal objects whose whole content is exhausted in their mutual relations. The "essence" of the numbers is completely expressed in their positions. And the concept of position must, first of all, be grasped in its greatest logical universality and scope. The distinctness required of the elements rests upon purely conceptual and not upon perceptual conditions. (15)

This position follows logically from the exclusion of quantity from number. If numbers are not quantitative, they cannot have qualitative differences, because the latter can only be found in numbers by reason of their quantity. As we saw earlier, three is what it is and an unum per se because of a formal quantitative order. It was the failure to consider this that gave force to the earlier objections which stated that number is only an accidental being. (16)

Therefore, if numbers have only an accidental unity, there is no problem of throwing a bridge between them and the continuum. In this problem we start out by supposing that number is heterogeneous and the continuum, homogeneous. If, now, number is only accidentally heterogeneous, that means that it is in se homogeneous. But then there is no opposition between number and the continuum: both are homogeneous. As was indicated earlier, what we have here is a "pseudo-problem" which arises from a failure to distinguish the terms involved. In this respect, it may be true to say that there is a certain continuity with the problem of Zeno. However, it should be remembered that Zeno played on the confusions in the objections of his opponents. The modern philosophers of mathematics cannot credit with such insight. They posit their own confusions.

By the logic of their position, the moderns are even forbidden to pose such problems as they do. If all heterogeneity is extrinsic, they cannot speak of any real differences in number. Thus, to distinguish even between "natural", "rational", irrational" numbers, can have no meaning. Many of the modern philosophers of mathematics hesitate to take this logical step. Most of them, while admitting that $2 = \frac{2}{1}$, insist that the first is a "natural" number and the second a "fractional" number. But what does all this circumlocution mean? If they are equal, they can have no intrinsic difference; if they are intrinsically different, they cannot be equal. Equality properly means identity in quantity.

This hesitancy is due to the fact that for most modern philosophers of mathematics, number has some sort of real heterogeneity. They seem to feel that there is something in the nature of number itself which makes it necessary to "generate" one number from another. Most practicing mathematicians find it difficult to follow the logic of Cassirer to its conclusion in idealism. One instance of this was given in the quotation from Mr. Smart, where he insisted on a quantitative basis for number. It is only with such a basis that "twoness" is specifically distinct from "threeness", and will thus require "generation". The very attempt by the moderns to reduce all mathematics to number or arithmetic is evidence of the fact that they conceive some sort of heterogeneity in number. But their attempts to explain this heterogeneity have ended by denying it. This confusion is partly due to a failure to appreciate the exact nature of this reduction.

The moderns have failed to see that there is a very great difference between the "rationalization" of continuous quantity by number and the "rationalization" of number by making it "continuous". The former can be carried on within the domain of mathematics, the latter cannot. When we reduce continuous quantity to the one or to number, we do so in order to know more clearly the quantity of the continuum. This is possible because of the greater formality and immateriality found in number as opposed to continuous quantity. We find a confusion in continuous quantity by reason of its potential divisibility. When, by measure, we reduce this confusion to the ratio of a number to a number or of a number to one, we have a "rationalization" of continuous quantity. This is universally recognized in our operations by which we apply number to the continuum. We know the distance from the earth to the moon better when we can express it as a certain number of miles. However, in the many distinct concepts necessary to grasp the distinction of number in se, we find an irrationality which will now call for a different sort of "rationalization". But neither of these rationalizations will have any meaning if there is no per se heterogeneity in number.

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NOTES

SECTION I

Chapter I

1. Continu et Discontinu, p. 195.

2. Ibid.

3. Ibid. pp. 193-194.

4. He cites Poincare, who, he says, held that "l'idee de l'ensemble des nombres naturels se reduit au principe de l'induction complete, qu'il considere comme jugement synthetique a priori dans le sens de Kant", and opposes him to Hermite, "qui, en ce qui concerne les concepts mathematiques, etait realiste platonicien." p. 194.

5. Ibid. p. 193.

6. Development of Mathematics, p. 12.

7. Concepts of the Calculus, p. 4.

8. Ibid. p. 267.

9. Cf. Bell, op. cit., p. 137, pp. 260 seq.; Pierpont, Mathematical Rigor.

10. The philosophical nature of this question is pointed out by Pierpont, op. cit.; Bell, op. cit.; and also in Fraenkel.

11. Op. cit., p. 200.

12. Op. cit., p. 23.

13. Those interested might consider carefully some of the erroneous opinions of Cornford, Wicksteed, Ross, Zeller, Van Pesch, Heiberg, et alia, who were simply incapable of comprehending the Greek mind, however good they may have been in paleography, language and historical research.

Chapter II

1. We shall not state the arguments in detail. Anyone interested can see Cajori, History of Zeno's Arguments, and especially Lee, Zeno of Elea.

2. Cf. Boyer and Bell, supra.

3. Cf. St Thomas' commentary on V Physics, lesson 5, nn. 1&2.

4. For the sharp distinction between mathematics and natural science cf. especially II Physics, 193 b 23. The definition of "between" is an exception, as will be pointed out.

5. IV Physics, 208 a 26 - 213 a 10.

6. Note that there is a difference between "changing continuously" and the "continuous", defined later.

7. For some of the modern attempts to define this term, vide Huntington, The Continuum, and Russell, Princ. of Math. and Int. to Math. Phil. pp. 12; 200-26; and 38-41, respectively.

8. Notice that this argument is dialectical. It merely points to the flaw in Zeno's reasoning. The same is true of the further refutations given in VI Physics, ch. 9, 239 b 5 seq. In Bk. VIII, ch. 8, 263 a 4 - 264 a 7, Aristotle points this out and then gives his own solution. We will quote the text later.

9. 263 a 4 - 265 b 9.

10. Cf. Strong, Procedures and Metaphysics, p. 249.

11. Also the Platonists who had strong leanings towards the Pythagorean number conceptions.

12. Heath, Hist. of Greek Math. p. 69, Vol. I.

Chapter III

1. Mead, C., Is Mathematics an Exact Science?

2. Cf. Bell, loc. cit.; Dedekind, Theory of Numbers, pp. 10-12; Hilbert, Foundations of Geometry, pp. 24-5; Russell, Princ. of Math., p. 190, and Int. to Math. Phil., p. 105. Lest

it be said that we are neglecting the "definitions" in some of the works cited, we must point out: a) the "definitions" suppose solutions of the problem which have been rejected by those who would insist on these same "definitions"; and b) an examination of the context in which these definitions occur reveals a supposition that we already know what is meant by "continuity" --- at least the "vague notion" familiar to everyone.

3. Cf. Bell, op. cit., p. 203.

4. Cf. the historical analyses of Bell, Boyer, Cajori, D.Smith.

5. He has a reference here to The Human Worth of Rigorous Thinking, N.Y. 1916, p. 277.

6. Cajori, Hist. of Math., p. 285. Nowhere in the book does he attempt to define either what he calls Aristotle's "sensuous, physical" continuum, or the one "created" by Cantor.

7. Cf. the commentary of St Thomas, V Metaphysics, less. 14, n. 977; also John of St Thomas, Curs. Phil., p. 541 a 4-15.

8. To be a species of quantity adds the nature of measure and measured. Cf. St Thomas, I Sent., D. 24, Q.1, a.3, ad 3; De Potentia, Q.9, a.7.

9. Cf. also Metaphysics, III, 997 b 15; VI, 1026 a 10; XI, 1064 b 10.

10. Cf. also II Physics, 193 b 30.

11. Cf. V Metaphysics, ch. 14, and St Thomas' commentary, less. 16, nn. 990-2.

12. Heath, Elements, p. 113, Vol. II.

13. Greek Math., p. 340, Vol. I. The terms "magnitude" and "of any assigned magnitude" are added by Heath and also by Hardie and Gaye in the Oxford tr. Actually, Aristotle is arguing here on the supposition of an infinite force or virtus in a finite magnitude.

14. Ibid. p. 344.

15. Elements, p. 234, Vol. I

16. Greek Math., pp. 221-5, Vol. I.

17. Ibid., p. 50, Vol. II.

18. Ibid., p. 17, Vol. I. He cites II Physics, 194 a 8; and Anal. Post., I, p. 76 a 22-25; p. 78 b 35-9.

19. Cf. Cajori, Bell, Smith, D.

Chapter IV

1. Ibid.

2. Berkeley, Analyst. Cited in Smith, D., Source Book in Math. p. 631.

3. Robins, Math. Tracts, p. 49, Vol. II.

4. Cf. Boyer, op. cit., pp. 248-9; Bell, op. cit., p. 287.

5. Op. cit., p. 32.

6. Cf. Boyer, op. cit., p. 270.

7. Ibid., p. 273.

8. Jourdain, Development of the Theory of Transfinite Numbers, article 1, p. 261.

9. Cajori, Hist. of Math., pp. 434-8.

10. Op. cit., p. 284.

11. Cf. Jourdain, op. cit., art. 2, pp. 303 seq.

12. Cf. Ibid.

13. Dedekind, op. cit., pp. 10 seq.

14. Ibid., p. 11.

15. Cantor, Grundlagen, p. 117.

16. Cf. Bell, Cajori, Boyer, Jourdain.

17. Cantor bases himself on Zeller and finds the arguments against the existence of the infinite in XI Metaphysics, ch. 10! St Thomas, commenting on the "petitio principii" given here, recognizes it as such and points out that the

argument is only dialectical, i.e. against a position. Cantor should have tried to refute the argument in III Physics, where there is a complete treatment of the infinite ex professo.

18. Cantor, op. cit., p. 115. He has a note there which says:

> The Procedure for the correct construction of concepts is in my opinion everwhere the same; one sets up a thing (Ding) having no properties, which at first is nothing more than a name or sign A and gives this according to a one law, different and even an infinite number of predicates, whose meaning is known through ideas already existent which may not contradict one another; in this way the relation of A to concepts already existent is determined; if the process is complete, then all the conditions for awakening the concept A (zur Weckung des Begriffes A) which was slumbering within us, are present and it emerges into being (Dasein), equipped with intra-subjective reality, which is all that can be demanded of concepts; to establish its transient meaning is then the business of metaphysics.

19. Math. Ann., p. 358, Vol. 17. (1880).

SECTION II

Chapter I

1. Cf. St. Thomas, Ia, Q.11, a.2, ad 2.

2. Cf. John of St Thomas, Curs. Theol., Vol. II, p. 108, n. 3.

3. Cf. St Thomas, De Veritate, Q.2, a.9, ad 10.

4. Curs. Theol., Vol. II, p. 50 a, n. 16.

> Quia mensura importat perfectionem, cum semper accipiatur pro mensura id quod perfectissimum est in unuoquoque genere; nec requiritur quod sit notificativum rei mensuratae, ut fundans imperfectam cognitionem: sed per modum alicujus magis simplicis et perfecti quo res mensurata magis ad unitatem et uniformitatem reducitur.

5. Ibid., p. 53 a, n. 22.

> Et quanto perfectior est mensura, tanto perfectius conjungitur suo mensurato, illudque magis ad se trahit quantum possibile est.

Chapter III

1. In De Sprituali̇bus Creaturis, a. 8, St Thomas calls the order found in the latter an ordo per accidens as opposed to the ordo per se found in a whole composed of per se ordered formal differences.

2. Russell, Int. to Math. Phil., p. 18.

3. Curs. Phil., Vol. 1, pp. 551 seq.

4. Cf. VII Metaphysics, ch. 13, 1039 a 1-15.

5. Ibid., p. 552.

> Et supponenda est vulgaris distinctio numeri numerantis et numeri numerati. Numerans dicitur ille, qui est ratio numerandi in intellectu, ut duo, tria, quatuor etc., quae sunt rationes, quibus omnem materiam numeramus. Numerus vero numeratus sunt res ipsae seu materia, quae numerationi isti subicitur. Et potest iste numerus numeratus sumi et generaliter pro omni mulittudine quomodocumque numerabili ab intellectu, etiam quae in rebus spiritualibus invenitur, et specialiter pro multitudine quantitativa, quae ratione quantitatis specialem rationem mensurae habet in numerando, ratione cuius aliquando ipsa discretio quantitatis respective ad substantiam, quae illi subicitur, dicitur numerus numerans, id est praebens rationem, qua numerabilis est quantitative.

6. Cf. also Curs. Theol., Vol. II, p. 103 a, n. 3.

7. This is the position of P. Merinerus, a disciple of Scotus.

8. Terrejon.

9. Cf. also the commentary of St Thomas, less. 4, nn. 1335 seq.

10. In V Metaph., less. 15; IV Metaph., less. 2, nn. 557 seq.; I Sent. D. 24, Q.1, a.3; and De Pot., Ia Pars, sup. cit.

11. Lesson 13, n. 1589:

> Et sic dualitas non erunt duae unitates, sed aliquid ex duabus unitatibus compositum. Aliter numerus non esset unum per se et vere, sed per accidens, sicut quae coacervantur.

12. Quodl. 10, a.1.

13. I Sent. D. 24, Q.1, a.1, ad 3.

> Numerus simpliciter est, cuius numerata faciunt aliquam aggregationem.

14. Lesson 17, nn. 1672-73:

> Quandoque enim ex multis fit compositio, ita quod totum compositum ex multis est unum quoddam, sicut domus composita ex suis partibus, et mixtum corpus ex elementis. Quandoque vero ex multis fit compositum, ita quod totum compositum non est unum simpliciter, sed solum secundum quid; sicut patet in cumulo m vel acervo lapidum, cum partes sunt in actu, cum non sint continuae. Unde simpliciter quidem est multa, sed solum secundum quid unum, prout ista multa associantur sibi in loco.
>
> Huius autem diversitatis ratio est, quia compositum quandoque sortitur speciem ab aliquo uno, quod est vel forma, ut patet in corpore mixto; vel compositio, ut patet in domo; vel ordo, ut patet in syllaba et numero. Et tunc oportet quod totum compositum sit unum simpliciter. Quandoque vero compositum sortitur speciem ab ipsa multitudine partium collectarum, ut patet in acervo et populo, et aliis hujusmodi: et in talibus totum compositum non est unum simpliciter, sed solum secundum quid.

15. Today, this would not be classed as an example but a metaphor.

Chapter III

1. Cf. St Thomas, In IV <u>Physics</u>, less. 17, n. 11; I Sent. D. 19, Q.2, a.1.

2. Cf. St Thomas, Ia, Q.30, a.1, ad 4; In VIII Metaph., less. 3, n. 1722; De Pot., Q.9, a.5, ad 6; ad 8; ad 9; Ibid., a.6; a.7; Q.Quodl., 10, a.1; In X Metaph., less. 4, nn. 1993 seq.

3. Cf. John of St Thomas, Curs. Phil., cited in footnote 5 of chapter II above.

4. Cf. St. Thomas, I Sent., D. 19, Q.1, a.1, ad 1; D. 17, Q.2, a.1, ad 2.

5. Cf. St Thomas, I Sent., D. 24, Q.1, a.2, ad 4.

Chapter IV

1. Frege's Logic, p. 489.

2. Ibid., pp.490-491.

3. Ibid., pp. 491-492.

4. Ibid., pp. 492-493.

5. Ibid., p. 494.

6. Ibid., p. 495.

7. Cf. St Thomas, De Pot., Q.9, a.7, c.

8. Cf. Int. to Math. Phil., sup. cit. The quotations which follow in the text are from ch. II.

9. Cohen and Nagel, Int. to Logic and Scientific Method, p. 150. The italics are theirs.

10. Peacock, Treatise on Algebra, p. 52, Vol. I.

11. Int. to Math. Phil., pp. 30-31.

12. Theory of Numbers, p. 68.

13. Princ. of Math., pp. 248-249.

14. Cassirer, Substance and Function, pp. 39 seq.

15. Ibid.

16. Cf. above, pp. 46 seq.; 51; 56.